4.7 Compound Interest

PREPARING FOR THIS SECTION
Before getting started, review the following:

- Simple Interest (Appendix, Section A.7, pp. 1010-1011)

Now work the 'Are You Prepared?' problems on page 322.

OBJECTIVES

1. Determine the Future Value of a Lump Sum of Money
2. Calculate Effective Rates of Return
3. Determine the Present Value of a Lump Sum of Money
4. Determine the Time Required to Double or Triple a Lump Sum of Money

1 Determine the Future Value of a Lump Sum of Money

Interest is money paid for the use of money. The total amount borrowed (whether by an individual from a bank in the form of a loan or by a bank from an individual in the form of a savings account) is called the principal. The rate of interest, expressed as a percent, is the amount charged for the use of the principal for a given period of time, usually on a yearly (that is, per annum) basis.

Simple Interest Formula

If a principal of \( P \) dollars is borrowed for a period of \( t \) years at a per annum interest rate \( r \), expressed as a decimal, the interest \( I \) charged is

\[
I = Prt \tag{1}
\]

Interest charged according to formula (1) is called simple interest.

In working with problems involving interest, we define the term payment period as follows:

Annual: Once per year
Monthly: 12 times per year
Semiannually: Twice per year
Quarterly: Four times per year

When the interest due at the end of a payment period is added to the principal so that the interest computed at the end of the next payment period is based on this new principal amount (old principal + interest), the interest is said to have been compounded. Compound interest is interest paid on principal and previously earned interest.

EXAMPLE 1

Computing Compound Interest

A credit union pays interest of 8% per annum compounded quarterly on a certain savings plan. If $1000 is deposited in such a plan and the interest is left to accumulate, how much is in the account after 1 year?

*Most banks use a 360-day “year.” Why do you think they do?
Solution  We use the simple interest formula, \( I = Prt \). The principal \( P \) is $1000 and the rate of interest is 8\% = 0.08. After the first quarter of a year, the time \( t \) is \( \frac{1}{4} \) year, so the interest earned is

\[
I = Prt = ($1000)(0.08)(\frac{1}{4}) = $20
\]

The new principal is \( P + I = $1000 + $20 = $1020 \). At the end of the second quarter, the interest on this principal is

\[
I = ($1020)(0.08)(\frac{1}{4}) = $20.40
\]

At the end of the third quarter, the interest on the new principal of $1020 + $20.40 = $1040.40 is

\[
I = ($1040.40)(0.08)(\frac{1}{4}) = $20.81
\]

Finally, after the fourth quarter, the interest is

\[
I = ($1061.21)(0.08)(\frac{1}{4}) = $21.22
\]

After 1 year the account contains \( $1061.21 + $21.22 = $1082.43 \).

The pattern of the calculations performed in Example 1 leads to a general formula for compound interest. To fix our ideas, let \( P \) represent the principal to be invested at a per annum interest rate \( r \) that is compounded \( n \) times per year, so the time of each compounding period is \( \frac{1}{n} \) years. (For computing purposes, \( r \) is expressed as a decimal.) The interest earned after each compounding period is given by formula (1).

\[
\text{Interest} = \text{principal} \times \text{rate} \times \text{time} = Pr \cdot \frac{1}{n} = P \cdot \left( \frac{r}{n} \right)
\]

The amount \( A \) after one compounding period is

\[
A = P + I = P + P \cdot \left( \frac{r}{n} \right) = P \cdot \left( 1 + \frac{r}{n} \right)
\]

After two compounding periods, the amount \( A \), based on the new principal \( P \cdot \left( 1 + \frac{r}{n} \right) \), is

\[
A = P \cdot \left( 1 + \frac{r}{n} \right) + P \cdot \left( 1 + \frac{r}{n} \right) \cdot \left( \frac{r}{n} \right) = P \cdot \left( 1 + \frac{r}{n} \right) \cdot \left( 1 + \frac{r}{n} \right) = P \cdot \left( 1 + \frac{r}{n} \right)^2
\]

A fter three compounding periods, the amount \( A \) is

\[
A = P \cdot \left( 1 + \frac{r}{n} \right)^2 + P \cdot \left( 1 + \frac{r}{n} \right)^2 \cdot \left( \frac{r}{n} \right) = P \cdot \left( 1 + \frac{r}{n} \right)^2 \cdot \left( 1 + \frac{r}{n} \right) = P \cdot \left( 1 + \frac{r}{n} \right)^3
\]
Exploration

To see the effects of compounding interest monthly on an initial deposit of $1, graph with

What is the future value of $1 in 30 years when the interest rate per annum is $(6\%)$? What is the future value of $1 in 30 years when the interest rate per annum is $(12\%)$?

Does doubling the interest rate double the future value?

$\frac{r}{12} = 0.12$ \quad $\frac{r}{12} = 0.06$

$Y_1 = a_1 + r_1 x \ldots x_{30}$.

Continuing this way, after $n$ compounding periods (1 year), the amount $A$ is

$$A = P \cdot \left(1 + \frac{r}{n}\right)^n$$

Because $t$ years will contain $n \cdot t$ compounding periods, after $t$ years we have

$$A = P \cdot \left(1 + \frac{r}{n}\right)^{nt}$$

**Theorem**

**Compound Interest Formula**

The amount $A$ after $t$ years due to a principal $P$ invested at an annual interest rate $r$ compounded $n$ times per year is

$$A = P \cdot \left(1 + \frac{r}{n}\right)^{nt} \quad (2)$$

For example, to rework Example 1, we would use $P = \$1000$, $r = 0.08$, $n = 4$ (quarterly compounding), and $t = 1$ year to obtain

$$A = P \cdot \left(1 + \frac{r}{n}\right)^{nt} = 1000 \left(1 + \frac{0.08}{4}\right)^4 = 1082.43$$

In equation (2), the amount $A$ is typically referred to as the **future value** of the account, while $P$ is called the **present value**.

**NOW WORK PROBLEM 3.**

**Comparing Investments Using Different Compounding Periods**

Investing $1000 at an annual rate of 10% compounded annually, semiannually, quarterly, monthly, and daily will yield the following amounts after 1 year:

Annual compounding ($n = 1$):\[ A = P \cdot (1 + r) = \$1000 \cdot (1 + 0.10) = \$1100.00 \]

Semiannual compounding ($n = 2$):\[ A = P \cdot \left(1 + \frac{r}{2}\right)^2 = \$1000 \cdot (1 + 0.05)^2 = \$1102.50 \]

Quarterly compounding ($n = 4$):\[ A = P \cdot \left(1 + \frac{r}{4}\right)^4 = \$1000 \cdot (1 + 0.025)^4 = \$1103.81 \]

Monthly compounding ($n = 12$):\[ A = P \cdot \left(1 + \frac{r}{12}\right)^{12} = \$1000 \cdot (1 + 0.00833)^{12} = \$1104.71 \]

Daily compounding ($n = 365$):\[ A = P \cdot \left(1 + \frac{r}{365}\right)^{365} = \$1000 \cdot (1 + 0.000274)^{365} = \$1105.16 \]

From Example 2 we can see that the effect of compounding more frequently is that the amount after 1 year is higher: $1000$ compounded 4 times a year at 10% results in $1103.81; 1000$ compounded 12 times a year at 10% results in $1104.71; and $1000$ compounded 365 times a year at 10% results in $1105.16. This leads to the
following question: What would happen to the amount after 1 year if the number of times that the interest is compounded were increased without bound?

Let’s find the answer. Suppose that $P$ is the principal, $r$ is the per annum interest rate, and $n$ is the number of times that the interest is compounded each year. The amount after 1 year is

$$A = P \left(1 + \frac{r}{n}\right)^n$$

Rewrite this expression as follows:

$$A = P \left(1 + \frac{r}{n}\right)^n = P \left(1 + \frac{1}{\frac{n}{r}}\right)^{n/r} = P \left[\left(1 + \frac{1}{\frac{n}{r}}\right)^{n/r}\right] = P \left[\left(1 + \frac{1}{h}\right)^h\right]$$

Now suppose that the number $n$ of times that the interest is compounded per year gets larger and larger; that is, suppose that $n \to \infty$. Then $h = \frac{n}{r} \to \infty$, and the expression in brackets equals $e$. [Refer to equation (2), page 317.] That is, $A \to Pe^r$.

Table 9 compares $\left(1 + \frac{r}{n}\right)^n$, for large values of $n$, to $e^r$ for $r = 0.05$, $r = 0.10$, $r = 0.15$, and $r = 1$. The larger that $n$ gets, the closer $\left(1 + \frac{r}{n}\right)^n$ gets to $e^r$.

No matter how frequent the compounding, the amount after 1 year has the definite ceiling $Pe^r$.

<table>
<thead>
<tr>
<th>$n$</th>
<th>$(1 + \frac{r}{n})^n$</th>
<th>$e^r$</th>
</tr>
</thead>
<tbody>
<tr>
<td>100</td>
<td>1.0512580</td>
<td>1.0512711</td>
</tr>
<tr>
<td>1000</td>
<td>1.0512698</td>
<td>1.051271</td>
</tr>
<tr>
<td>10,000</td>
<td>1.0512710</td>
<td>1.051271</td>
</tr>
<tr>
<td>$e^r$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

When interest is compounded so that the amount after 1 year is $Pe^r$, we say the interest is **compounded continuously**.

**Theorem**

**Continuous Compounding**

The amount $A$ after $t$ years due to a principal $P$ invested at an annual interest rate $r$ compounded continuously is

$$A = Pe^{rt}$$

**Example 3**

**Using Continuous Compounding**

The amount $A$ that results from investing a principal $P$ of $1000$ at an annual rate $r$ of $10\%$ compounded continuously for a time $t$ of 1 year is

$$A = 1000e^{0.10} = (1000)(1.10517) = 1105.17$$

Now work Problem 11.
**Section 4.7 Compound Interest**

**Exploration**

For the IRA described in Example 4, how long will it be until $6000?

[Hint: Graph $Y_1 = 2000e^{0.07x}$ and $Y_2 = 4000$. Use INTERSECT to find $x$.]

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**Calculate Effective Rates of Return**

The **effective rate of interest** is the equivalent annual simple rate of interest that would yield the same amount as compounding after 1 year. For example, based on Example 3, a principal of $1000 will result in $1105.17 at a rate of 10% compounded continuously. To get this same amount using a simple rate of interest would require that interest of $1105.17 − $1000.00 = $105.17 be earned on the principal. Since $105.17 is 10.517% of $1000, a simple rate of interest of 10.517% is needed to equal 10% compounded continuously. The effective rate of interest of 10% compounded continuously is 10.517%.

Based on the results of Examples 2 and 3, we find the following comparisons:

<table>
<thead>
<tr>
<th>Method</th>
<th>Annual Rate</th>
<th>Effective Rate</th>
</tr>
</thead>
<tbody>
<tr>
<td>Annual compounding</td>
<td>10%</td>
<td>10%</td>
</tr>
<tr>
<td>Semiannual compounding</td>
<td>10%</td>
<td>10.25%</td>
</tr>
<tr>
<td>Quarterly compounding</td>
<td>10%</td>
<td>10.381%</td>
</tr>
<tr>
<td>Monthly compounding</td>
<td>10%</td>
<td>10.471%</td>
</tr>
<tr>
<td>Daily compounding</td>
<td>10%</td>
<td>10.516%</td>
</tr>
<tr>
<td>Continuous compounding</td>
<td>10%</td>
<td>10.517%</td>
</tr>
</tbody>
</table>

**EXAMPLE 4**

**Computing the Effective Rate of Interest**

On January 2, 2004, $2000 is placed in an Individual Retirement Account (IRA) that will pay interest of 7% per annum compounded continuously.

(a) What will the IRA be worth on January 1, 2024?

(b) What is the effective rate of interest?

**Solution**

(a) The amount $A$ after 20 years is

$$A = Pe^{rt} = $2000e^{0.07(20)} = $8110.40$$

(b) First, we compute the interest earned on $2000 at $r = 7\%$ compounded continuously for 1 year.

$$A = $2000e^{0.07(1)}$$

$$= $2145.02$$

So the interest earned is $2145.02 − $2000.00 = $145.02. Use the simple interest formula $I = Prt$, with $I = $145.02, $P = $2000$, and $r = 1$, and solve for $r$, the effective rate of interest.

$$r = \frac{145.02}{2000} = 0.07251$$

The effective rate of interest is 7.251%.

**NOW WORK PROBLEM 23.**
Determine the Present Value of a Lump Sum of Money

When people engaged in finance speak of the "time value of money," they are usually referring to the present value of money. The present value of A dollars to be received at a future date is the principal that you would need to invest now so that it will grow to A dollars in the specified time period. The present value of money to be received at a future date is always less than the amount to be received, since the amount to be received will equal the present value (money invested now) plus the interest accrued over the time period.

We use the compound interest formula (2) to get a formula for present value. If $P$ is the present value of A dollars to be received after $t$ years at a per annum interest rate $r$ compounded $n$ times per year, then, by formula (2),

$$A = P \cdot \left(1 + \frac{r}{n}\right)^{nt}$$

To solve for $P$, we divide both sides by $\left(1 + \frac{r}{n}\right)^{nt}$. The result is

$$\frac{A}{\left(1 + \frac{r}{n}\right)^{nt}} = P \text{ or } P = A \cdot \left(1 + \frac{r}{n}\right)^{-nt}$$

**Theorem**

**Present Value Formulas**

The present value $P$ of A dollars to be received after $t$ years, assuming a per annum interest rate $r$ compounded $n$ times per year, is

$$P = A \cdot \left(1 + \frac{r}{n}\right)^{-nt} \quad (5)$$

If the interest is compounded continuously, then

$$P = Ae^{-ri} \quad (6)$$

To prove (6), solve formula (4) for $P$.

**EXAMPLE 5**

**Computing the Value of a Zero-Coupon Bond**

A zero-coupon (noninterest-bearing) bond can be redeemed in 10 years for $1000. How much should you be willing to pay for it now if you want a return of

(a) 8% compounded monthly?

(b) 7% compounded continuously?

**Solution**

(a) We are seeking the present value of $1000. We use formula (5) with $A = 1000$, $n = 12$, $r = 0.08$, and $t = 10$.

$$P = A \cdot \left(1 + \frac{r}{n}\right)^{-nt} = 1000 \cdot \left(1 + \frac{0.08}{12}\right)^{-12(10)} = 450.52$$

For a return of 8% compounded monthly, you should pay $450.52 for the bond.
(b) Here we use formula (6) with \( A = 1000, r = 0.07 \), and \( t = 10 \).

\[
P = A e^{-rt} = 1000e^{-(0.07)(10)} = 496.59
\]

For a return of 7% compounded continuously, you should pay $496.59 for the bond.

NOW WORK PROBLEM 13.

**EXAMPLE 6**  
Rate of Interest Required to Double an Investment

What annual rate of interest compounded annually should you seek if you want to double your investment in 5 years?

**Solution**  
If \( P \) is the principal and we want \( P \) to double, the amount \( A \) will be \( 2P \). We use the compound interest formula with \( n = 1 \) and \( t = 5 \) to find \( r \).

\[
A = P \cdot \left( 1 + \frac{r}{n} \right)^{nt} = 2P
\]

\[
2 = (1 + r)^5
\]

Take the fifth root of each side.

\[
r = \sqrt[5]{2} - 1 \approx 0.148698
\]

The annual rate of interest needed to double the principal in 5 years is 14.87%.

NOW WORK PROBLEM 25.

**EXAMPLE 7**  
Doubling and Tripling Time for an Investment

(a) How long will it take for an investment to double in value if it earns 5% compounded continuously?

(b) How long will it take to triple at this rate?

**Solution**  
(a) If \( P \) is the initial investment and we want \( P \) to double, the amount \( A \) will be \( 2P \). We use formula (4) for continuously compounded interest with \( r = 0.05 \). Then

\[
A = Pe^{rt}
\]

\[
2P = Pe^{0.05t} \quad A = 2P, r = 0.05
\]

Cancel the \( P \)’s.

\[
2 = e^{0.05t}
\]

Rewrite as a logarithm.

\[
0.05t = \ln 2 \quad \text{Solve for } t.
\]

\[
t = \frac{\ln 2}{0.05} \approx 13.86
\]

It will take about 14 years to double the investment.
(b) To triple the investment, we set \( A = 3P \) in formula (4).

\[
A = Pe^{rt}
\]
\[
3P = Pe^{0.05t}
\]
\[
3 = e^{0.05t}
\]

Cancel the \( P \) ’s.

\[
0.05t = \ln 3
\]

Rewrite as a logarithm.

\[
t = \frac{\ln 3}{0.05} \approx 21.97
\]

Solve for \( t \).

It will take about 22 years to triple the investment.

NOW WORK PROBLEM 31.