Formulas for Liapunov Functions for Systems of Linear Difference Equations

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Abstract

Explicit quadratic Liapunov functions that provide necessary and sufficient conditions for the asymptotic stability of the system of linear difference equations $x(t+1) = Ax(t)$ are constructed by transforming the original systems to $y(t+1) = Gy(t)$, where $G$ is a companion matrix associated with the characteristic polynomial of $A$. A necessary and sufficient condition for all roots of the characteristic polynomial to lie in the unit circle $|z| < 1$ on the complex plane is also derived.

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1 Introduction

In this paper, we derive formulas for Liapunov functions for the determination of asymptotic stability of solutions of the system of linear difference equations

\[ x(t+1) = Ax(t) \]  

(1.1)

where \( A = (a_{ij})_{n \times n} \) is an \( n \times n \) matrix and \( x(t) \) is an \( n \)-vector function of the time \( t \). Using these formulas, we also derive a new set of necessary and sufficient conditions for all roots of the characteristic polynomial to lie in the unit circle \(|z| < 1\) on the complex plane.

The theory of discrete dynamical systems has grown tremendously in the last decade. The growth has been strongly promoted by the advanced technology in scientific computation and the large number of applications to models in biology, engineering, and other physical sciences. It is the stability and asymptotic behavior of solutions of these models that is especially important to many investigators. The stability of a discrete process is the ability of the process to resist a priori unknown influences (small). A process is said to be stable if such disturbances do not change it. This property turns out to be of utmost importance since, in general, an individual predictable process can be physically realized only if it is stable in the corresponding natural sense. For the historical background, fundamental theorems on stability, and discussion on applications, we refer the reader, for example, to the works of Elaydi [5], Gajić and Qureshi [6], Hahn and Parks [7], Jury [8], Kelly and Peterson [10], and Kocic and Ladas [11].

One of the most powerful methods used in stability theory is due to A. M. Liapunov [16]. Liapunov’s method consists in the use of an auxiliary function, the Liapunov function, which generalizes the role of the energy function in mechanical systems. It requires no knowledge of solutions near the solution under study; that is, the assertion of stability is made by investigating the equation itself and not by finding the solutions of the equation. However, another price has to be paid: the application of Liapunov’s method demands the construction of Liapunov functions. Such constructions are generally difficult and have been made only for certain classes of equations. If a Liapunov function is found, then we may gain, in addition to stability, further information about solutions such as boundedness, convergence, existence of
periodic solutions, regions of contraction, and estimates of the influence of constantly acting perturbations to the system.

A general approach to studying nonlinear problems is through Liapunov functions for linear equations (see Barbashin [1]). Such approach has been very successful for differential systems for a quite long time. However, its extension to discrete dynamics is still in early stages, and it appears that there is a great need for careful analysis of specific simple equations as a guide to the development of the general theory. Our goal here is to provide a unified approach to construct Liapunov functions for linear system (1.1). Explicit formulas for Liapunov functions for (1.1) will be derived in Section 2. Stability criteria for the characteristic equation of (1.1) will be given in Section 3. We leave the discussion of nonlinear equations for future investigations.

2 Formulas for Liapunov Functions.

For $x \in \mathbb{R}^n$, $|\cdot|$ denotes the Euclidean norm of $x$. We denote the determinant of an $n \times n$ matrix $B$ by $|B|$ and the norm of $B$ by $\|B\| = \sup \{|Bx| : |x| \leq 1\}$. An asterisk will denote the conjugate transpose of a matrix; that is, $B^* = (\bar{b}_{ji})_{n \times n}$ when $B = (b_{ij})_{n \times n}$. If $x$, or its components, is written without its argument, then the argument is $t$.

Let $t_0$ be any real number and define

$$N(t_0) = \{t_0, t_0 + 1, t_0 + 2, \ldots, t_0 + k, \ldots\}.$$ 

A function $x : N(t_0) \rightarrow \mathbb{R}^n$ is said to be a solution of (1.1) with initial condition $(t_0, x_0)$ if $x(t)$ satisfies (1.1) for $t \geq t_0$ and $x(t_0) = x_0$. We denote such a solution by $x(t) = x(t, t_0, x_0)$. The solution $x = 0$ is asymptotically stable if $x(t, t_0, x_0) \rightarrow 0$ as $t \rightarrow \infty$ for each $(t_0, x_0) \in \mathbb{R} \times \mathbb{R}^n$. The solution can be found by calculating the power of $A$. In fact,

$$x(t) = A^{(t-t_0)}x_0.$$ 

Thus, the zero solution of (1.1) is asymptotically stable if and only if $|\lambda| < 1$ for each eigenvalue $\lambda$ of $A$. An alternative criterion for asymptotic stability is to pose conditions on
the coefficients of the characteristic polynomial of $A$

$$\triangle(\lambda) = |\lambda I - A| = \lambda^n + p_1\lambda^{n-1} + \cdots + p_{n-1}\lambda + p_n$$

(2.1)

so that all roots of (2.1) lie within the unit circle on the complex plane. Early literature on this subject includes the work of Cohn [3] and Schur [18]. We present a criterion due to Jury [8] in connection with our investigation. The following array of numbers is constructed with $p_0 = 1$.

\[
\begin{array}{cccccccc}
  p_0 & p_1 & p_2 & \cdots & p_{n-2} & p_{n-1} & p_n \\
  p_n & p_{n-1} & p_{n-2} & \cdots & p_2 & p_1 & p_0 \\
  q_0 & q_1 & q_2 & \cdots & q_{n-2} & q_{n-1} & 0 \\
  q_{n-1} & q_{n-2} & q_{n-3} & \cdots & q_1 & q_0 & 0 \\
  r_0 & r_1 & r_2 & \cdots & r_{n-2} & 0 & 0 \\
  r_{n-2} & r_{n-3} & r_{n-4} & \cdots & r_0 & 0 & 0 \\
  \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
  s_0 & s_1 & s_2 & \cdots & 0 & 0 & 0 \\
  s_2 & s_1 & s_0 & \cdots & 0 & 0 & 0 \\
  u_0 & u_1 & 0 & \cdots & 0 & 0 & 0 \\
  u_1 & u_0 & 0 & \cdots & 0 & 0 & 0 \\
\end{array}
\]

The first two rows consist of the coefficients of the characteristic polynomial of $A$ “forward” in the first row and “backward” in the second row. The elements $q_k$ of the third row are calculated in the form

$$q_k = \begin{vmatrix} p_0 & p_{n-k} \\ p_n & p_k \end{vmatrix} \quad (k = 0, 1, \ldots, n - 1)$$

so that the third row consists of $n$ elements. The fourth row consists of the elements $q_k$ of the third row written in reverse order. One continues the construction of the array in a similar way so that, for example, the elements $r_k$ of the fifth row are calculated as

$$r_k = \begin{vmatrix} q_0 & q_{n-1-k} \\ q_{n-1} & q_k \end{vmatrix} \quad (k = 0, 1, \ldots, n - 2) \quad \text{and} \quad u_k = \begin{vmatrix} s_0 & s_{2-k} \\ s_{2-k} & s_0 \end{vmatrix} \quad (k = 0, 1).$$

There are $2n$ rows in the array.

**Theorem A ([8]).** A necessary and sufficient condition for all roots of the characteristic polynomial of $A$ to lie within the unit circle $|z| < 1$ in the complex plane is that the numbers
\(q_0, r_0, \ldots, s_0, u_0\) are positive,
\[
\sum_{i=0}^{n} p_i > 0, \quad \text{and} \quad \sum_{i=0}^{n} (-1)^i p_i > 0.
\]

Other criteria for the stability of the characteristic equation of (1.1) will be discussed in Section 3. For \(n = 2\), Jury’s array takes the form
\[
\begin{array}{ccc}
1 & p_1 & p_2 \\
p_2 & p_1 & 1 \\
1 - p_2^2 & p_1(1 - p_2) & 0 \\
p_1(1 - p_2) & 1 - p_2^2 & 0
\end{array}
\]
Thus, all roots of the polynomial \(\lambda^2 + p_1\lambda + p_2\) lie within the unit circle \(|z| < 1\) if and only if
\[
1 + p_1 + p_2 > 0, \quad 1 - p_1 + p_2 > 0, \quad \text{and} \quad 1 - p_2 > 0.
\] (2.2)

For \(n = 3\), Jury’s array yields that all roots of the polynomial \(\lambda^3 + p_1\lambda^2 + p_2\lambda + p_3\) lie within the unit circle \(|z| < 1\) if and only if
\[
1 + p_1 + p_2 + p_3 > 0, \quad 1 - p_1 + p_2 - p_3 > 0, \quad \text{and} \quad 1 - p_2^2 > |p_2 - p_1|p_3|.
\] (2.3)

The third characterization of the asymptotic stability of (1.1) is to construct a quadratic function \(V(x) = x^T B x\) where \(B = (b_{ij})_{n \times n}\) is symmetric and \(x \in \mathbb{R}^n\). Let \(x(t) = x(t, t_0, x_0)\) be a solution of (1.1). Then
\[
V(x(t+1)) - V(x(t)) = V(Ax) - V(x) = -x^T C x
\]
where \(C = (c_{ij})_{n \times n}\). For a given \(n \times n\) symmetric matrix \(C\), this leads to the equation
\[
A^T BA - B = -C
\] (2.4)
for an unknown matrix \(B\). If there are positive definite matrices \(B\) and \(C\) such that (2.4) is satisfied, then the zero solution of (1.1) is asymptotically stable (see Lakshmikantham and Trigiante [12, p.112]).

It is also well-known that for a given positive definite matrix \(C\), the matrix \(B\) in (2.4) is positive definite if and only if all eigenvalues \(\lambda\) of \(A\) satisfy \(|\lambda| < 1\) (see Lancaster and
Tismenetsky [13, p.451]). Thus, determining the asymptotic stability of (1.1) is equivalent to solving the matrix equation (2.4) for a positive definite $B$ or finding the positive definite quadratic form $x^TBx$ when $C$ is positive definite. The existence of $B$ is known. However, the structure of $B$ is not well understood, and this presents a significant challenge in applications. In this section, we will derive a formula for $V(x) = x^TBx$ with $V(Ax) - V(x)$ negative semi-definite along the solutions of (1.1) and $B$ having a much simpler structure. Such work has been done for the system of linear ordinary differential equations $x' = Ax(t)$. We follow the work of Smith [19], Tsai [20], and Zhang [21] and adopt the notations there. The following invariance principle is well-known (see LaSalle [14]), and we state a simple version of it here for later reference.

**Theorem B.** ([14]) If there exists a positive definite quadratic Liapunov function $V$ on $\mathbb{R}^n$ with $V(Ax) - V(x) \leq 0$, then each solution $x(t, t_0, x_0)$ of (1.1) is bounded and approaches the set $M$ as $t \to \infty$, where $M$ is the maximal invariant subset of

$$E = \{x \in \mathbb{R}^n \mid V(Ax) - V(x) = 0\}$$

with respect to system (1.1).

Notice that the sign-definiteness of $V(Ax) - V(x)$ is not required in Theorem B. This fact turns out to be very important because almost any attempt to construct a Liapunov function for a nonlinear system leads to a function which is not sign-definite (see Barbashin [1] or LaSalle [14]).

Now consider the $n \times n$ matrix

$$G = \begin{pmatrix} -p_1 & -1 & 0 & 0 & \cdots & 0 & \cdots & 0 \\ p_2 & 0 & -1 & 0 & \cdots & 0 & \cdots & 0 \\ -p_3 & 0 & 0 & -1 & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ (-1)^k p_k & 0 & 0 & 0 & \cdots & -1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ (-1)^{n-1} p_{n-1} & 0 & 0 & 0 & \cdots & 0 & \cdots & -1 \\ (-1)^n p_n & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 \end{pmatrix} \quad (2.5)$$
where \( p_k \) is defined in (2.1). A direct computation shows

\[
|\lambda I - G| = \lambda^n + p_1 \lambda^{n-1} + p_2 \lambda^{n-2} + \cdots + p_{n-1} \lambda + p_n.
\]

This implies that the characteristic polynomial of \( G \) is the same as that of \( A \). For \( n = 4 \), we have

\[
G = \begin{pmatrix}
-p_1 & -1 & 0 & 0 \\
p_2 & 0 & -1 & 0 \\
-p_3 & 0 & 0 & -1 \\
p_4 & 0 & 0 & 0
\end{pmatrix}.
\]

Next, consider the system of difference equations

\[
y(t + 1) = G y(t)
\]

(2.6)

where \( y(t) = (y_1(t), y_2(t), \ldots, y_n(t))^T \). Again, for \( n = 4 \), (2.6) becomes

\[
\begin{align*}
y_1(t + 1) &= -p_1 y_1(t) - y_2(t) \\
y_2(t + 1) &= p_2 y_1(t) - y_3(t) \\
y_3(t + 1) &= -p_3 y_1(t) - y_4(t) \\
y_4(t + 1) &= p_4 y_1(t)
\end{align*}
\]

We now introduce the determinants \( M_k \) (Smith [19]) which play a role similar to the \( \triangle_k \)'s in the Routh-Hurwitz stability criterion for systems of linear ordinary differential equations (see Burton [2, p.60] and Zhang [21]). Define

\[
M_1 = \begin{vmatrix}
p_0 & p_1 + p_{-1} \\
p_1 & p_2 + p_0
\end{vmatrix},
\quad M_2 = \begin{vmatrix}
p_0 & p_1 + p_{-1} & p_2 + p_{-2} \\
p_1 & p_2 + p_0 & p_3 + p_{-1} \\
p_2 & p_3 + p_1 & p_4 + p_0
\end{vmatrix},
\quad \cdots
\]

\[
M_n = \begin{vmatrix}
p_0 & p_1 + p_{-1} & p_2 + p_{-2} & \cdots & p_n + p_{-n} \\
p_1 & p_2 + p_0 & p_3 + p_{-1} & \cdots & p_{1+n} + p_{1-n} \\
p_2 & p_3 + p_1 & p_4 + p_0 & \cdots & p_{2+n} + p_{2-n} \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
p_n & p_{n+1} + p_{n-1} & p_{n+2} + p_{n-2} & \cdots & p_{2n} + p_0
\end{vmatrix}
\]

(2.7)

where the \( p_k \) are defined in (2.1) with \( p_0 = 1 \) and \( p_\nu = 0 \) for all \( \nu < 0 \) or \( \nu > n \).
An explicit linear function $L$ that transforms (1.1) to (2.6) will be derived later. We now prove the following theorem which is essential for the construction of Liapunov functions for (1.1).

**Theorem 2.1.** Let $V(y) = (y_1, y_2, \cdots, y_n)^T$ be the quadratic form obtained by replacing the first row of $M_{n-1}$ by the vector

\[
\left( \sum_{\alpha-\beta=0} y_\alpha y_\beta, -\sum_{\alpha-\beta=\pm 1} y_\alpha y_\beta, \sum_{\alpha-\beta=\pm 2} y_\alpha y_\beta, \cdots, (-1)^{n-1} \sum_{\alpha-\beta=\pm (n-1)} y_\alpha y_\beta \right).
\]

That is

\[
V(y) = \left| \begin{array}{cccc}
\sum_{\alpha-\beta=0} y_\alpha y_\beta & -\sum_{\alpha-\beta=\pm 1} y_\alpha y_\beta & \sum_{\alpha-\beta=\pm 2} y_\alpha y_\beta & \cdots & (-1)^{n-1} \sum_{\alpha-\beta=\pm (n-1)} y_\alpha y_\beta \\
p_1 & p_2 + p_0 & p_3 + p_{-1} & \cdots & p_n + p_{2-n} \\
p_2 & p_3 + p_1 & p_4 + p_0 & \cdots & p_{1+n} + p_{3-n} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
p_{n-1} & p_n + p_{n-2} & p_{n+1} + p_{n-3} & \cdots & p_{2(n-1)} + p_0
\end{array} \right|
\]

where $C_k$ is the cofactor of the element in the first row and $k$-th column of $M_{n-1}$. Then

\[
V(Gy) - V(y) = y^T [G^T B G - B] y = -M_n y_1^2.
\] (2.9)

**Proof.** We first solve the matrix equation

\[
G^T B G - B = -\text{diag}(M_n, 0, \cdots, 0) =: -C
\] (2.10)

for a symmetric matrix $B = (b_{\alpha\beta})_{n \times n}$. Smith [19] provides the following representation for $B$ when $|\lambda| < 1$ for every eigenvalue $\lambda$ of $A$:

\[
B = \frac{1}{2\pi i} \oint (G^T - Iz^{-1})^{-1} C (G - Iz)^{-1} z^{-1} dz
\]

where $\oint$ denotes integration in the counterclockwise direction around the circle $|z| = 1$ in the complex $z$ plane and $i = \sqrt{-1}$ is the usual imaginary unit of the complex number system.
Since \((z)^{-1} = z\) for \(|z| = 1\) and 
\[(Iz^{-1} - G^T)^{-1} = [(Iz^{-1} - G)^*]^{-1} = [(Iz - G)^{-1}]^*,\]
it follows that \(B\) is Hermitian and 
\[B = \frac{1}{2\pi i} \oint [(Iz - G)^{-1}]^* C(Iz - G)^{-1}z^{-1}dz\]
For \(\Delta(\lambda) = |\lambda I - G| \neq 0\), the formula for the inverse of a matrix gives 
\[(\lambda I - G)^{-1} = \frac{1}{\Delta(\lambda)}(\Delta_{\alpha\beta}(\lambda))_{n \times n}\]
where \(\Delta_{\alpha\beta}\) is the cofactor of the element in the \(\alpha\)-th column and \(\beta\)-th row of \(\lambda I - G\). Observe that 
\[[ (Iz - G)^{-1} ]^* = \left[ \frac{1}{\Delta(z)}(\Delta_{\alpha\beta}(z))_{n \times n} \right]^* = \frac{1}{\Delta(z^{-1})}(\Delta_{\alpha\beta}(z^{-1}))^T_{n \times n}\]
and 
\[(\Delta_{\alpha\beta}(z^{-1}))^T_{n \times n} C(\Delta_{\alpha\beta}(z))_{n \times n}
\begin{align*}
&= M_n(\Delta_{11}(z^{-1}), \Delta_{12}(z^{-1}), \cdots, \Delta_{1n}(z^{-1}))^T (\Delta_{11}(z), \Delta_{12}(z), \cdots, \Delta_{1n}(z)) \\
&= M_n(\Delta_{1\alpha}(z^{-1})\Delta_{1\beta}(z))_{n \times n}.
\end{align*}\]
Thus,
\[B = \frac{M_n}{2\pi i} \oint \frac{((\Delta_{1\alpha}(z^{-1})\Delta_{1\beta}(z))_{n \times n})z^{-1}dz}{\Delta(z^{-1})\Delta(z)}\]
By the definition of \(\Delta_{\alpha\beta}(\lambda)\), we have \(\Delta_{1k}(\lambda) = (-1)^{1+k}(\lambda)^{n-k}\) for \(k = 1, 2, \cdots, n\), and hence 
\[\Delta_{1\alpha}(z^{-1})\Delta_{1\beta}(z) = (-1)^{1+\alpha}(z^{-1})^{n-\alpha}(-1)^{1+\beta}(z)^{n-\beta} = (-1)^{\alpha+\beta}z^{\alpha-\beta}.\]
This yields 
\[b_{\alpha\beta} = \frac{M_n(-1)^{\alpha+\beta}}{2\pi i} \oint \frac{z^{\alpha-\beta-1}}{\Delta(z^{-1})\Delta(z)}dz \quad (2.11)\]
where \(B = (b_{\alpha\beta})_{n \times n}\). We now define 
\[\Omega_\mu = \frac{1}{2\pi i} \oint \frac{z^{\mu-1}}{\Delta(z^{-1})\Delta(z)}dz\]
for \( \mu = 0, \pm 1, \pm 2, \cdots, \pm n \). Smith [19] showed that
\[
\Omega_\mu = \Omega_{-\mu} = M_n^{-1} \phi(\mu)
\]
for \( \mu = 0, 1, 2, \cdots, n \), where \( \phi(\mu) \) is the cofactor of the element in the first row and \((\mu + 1)\)-th column of the determinant \( M_n \). Therefore,
\[
b_{\alpha\beta} = (-1)^{\alpha + \beta} \phi(\alpha - \beta)
\]
for all \( \alpha, \beta = 1, 2, \cdots, n \). Notice that if \( \alpha - \beta = k \), then
\[
(-1)^{\alpha + \beta} \phi(\alpha - \beta) = (-1)^{2\beta + k} \phi(k) = (-1)^k \phi(k).
\]
Thus, \( b_{\alpha\beta} = (-1)^k \phi(k) \) for \( \alpha - \beta = k \) and
\[
B = \begin{pmatrix}
b_{11} & b_{12} & b_{13} & \cdots & b_{1(n-1)} & b_{1n} \\
b_{12} & b_{11} & b_{12} & \cdots & b_{1(n-2)} & b_{1(n-1)} \\
b_{13} & b_{12} & b_{11} & \cdots & b_{1(n-3)} & b_{1(n-2)} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
b_{1(n-1)} & b_{1(n-2)} & b_{1(n-3)} & \cdots & b_{11} & b_{12} \\
b_{1n} & b_{1(n-1)} & b_{1(n-2)} & \cdots & b_{12} & b_{11}
\end{pmatrix}.
\tag{2.12}
\]

For \( n = 4 \), we have
\[
B = \begin{pmatrix}
b_{11} & b_{12} & b_{13} & b_{14} \\
b_{12} & b_{11} & b_{12} & b_{13} \\
b_{13} & b_{12} & b_{11} & b_{12} \\
b_{14} & b_{13} & b_{12} & b_{11}
\end{pmatrix}
\]
It is clear that we only need to know \( b_{1\beta} \) for \( \beta = 1, 2, \cdots, n \). Since the last column of \( M_n \) is given by \((p_n, 0, \cdots, 0, 1)^T\), we see that the cofactor of the element in the first row and \((\mu + 1)\)-th column of \( M_n \) is the same as that of \( M_{n-1} \) for \( \mu = 0, 1, \cdots, n - 1 \). Thus, we find that
\[
V(y) = y^T B y
\]
is the quadratic form obtained by replacing the first row of \( M_{n-1} \) by the vector
\[
\begin{pmatrix}
\sum_{\alpha-\beta=0} y_{\alpha} y_{\beta}, & -\sum_{\alpha-\beta=\pm 1} y_{\alpha} y_{\beta}, & \sum_{\alpha-\beta=\pm 2} y_{\alpha} y_{\beta}, & \cdots, & (-1)^{n-1} \sum_{\alpha-\beta=\pm (n-1)} y_{\alpha} y_{\beta}
\end{pmatrix},
\]
and
\[
y^T (G^T B G - B) y = -M_n y_1^2.
\]
It has now been proved that (2.8) satisfies (2.9) provided that $M_n > 0$. To remove this assumption, observe that when (2.11) is substituted into (2.10), both sides are polynomials in $p_1, p_2, \cdots, p_n$. This completes the proof.

**Example 2.1.** The following formulas can be directly verified and will be used later. For $n = 2$,

$$V(y) = \begin{vmatrix} y_1^2 + \frac{y_2^2}{p_1} & -2y_1y_2 \end{vmatrix} = (p_2 + 1)[y_1^2 + y_2^2] + 2p_1y_1y_2$$

and

$$V(Gy) - V(y) = -M_2y_1^2$$

where $M_2 = (1 - p_2)(1 + p_1 + p_2)(1 - p_1 + p_2)$. If $1 + p_2 \neq 0$, which is the case if (2.2) holds, then

$$V(y) = \frac{1}{1 + p_2}(1 + p_1 + p_2)(1 - p_1 + p_2)y_1^2$$

$$+ \frac{1}{1 + p_2}[p_1y_1 + (1 + p_2)y_2]^2.$$

If (2.2) is satisfied, then $V(y)$ is positive definite, $M_2 > 0$, and hence the zero solution of (2.6) for $n = 2$ is asymptotically stable by Theorem B.

For $n = 3$,

$$V(y) = \begin{vmatrix} \sum_{\alpha-\beta=0}^\infty y_\alpha y_\beta & -\sum_{\alpha-\beta=\pm 1}^\infty y_\alpha y_\beta & \sum_{\alpha-\beta=\pm 2}^\infty y_\alpha y_\beta \\ p_1 & p_2 + 1 & p_3 \\ p_2 & p_3 + p_1 & 1 \end{vmatrix}$$

$$= (1 - p_3^2 + p_2 - p_1p_3)(y_1^2 + y_2^2 + y_3^2) + 2(p_1 - p_2p_3)(y_1y_2 + y_2y_3)$$

$$+ 2[p_1^2 - p_2^2 - (p_2 - p_1p_3)]y_1y_3$$

and

$$V(Gy) - V(y) = -M_3y_1^2$$

where $M_3 = \sum_{i=0}^3 p_i \sum_{i=0}^3 (-1)^ip_i[(1 - p_3^2) - (p_2 - p_1p_3)]$. 

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Note that $V(y)$ can be written as

$$V(y) = (y_1, y_2, y_3) \begin{pmatrix} a & b & c \\ b & a & b \\ c & b & a \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

where $a = 1 - p_3^2 + p_2 - p_1 p_3$, $b = p_1 - p_2 p_3$, and $c = p_1^2 - p_2^2 - (p_2 - p_1 p_3)$. Hence $V(y)$ is positive definite if and only if the matrix

$$A = \begin{pmatrix} a & b & c \\ b & a & b \\ c & b & a \end{pmatrix}$$

is positive definite. It is known that $A$ is positive definite if and only if

$$a > 0, \quad \left| \begin{array}{cc} a & b \\ b & a \end{array} \right| = a^2 - b^2 > 0, \quad \text{and} \quad \left| \begin{array}{ccc} a & b & c \\ b & a & b \\ c & b & a \end{array} \right| = (a - c)(a(a + c) - 2b^2) > 0.$$ 

We claim that if (2.3) holds, then the above three conditions are satisfied. Clearly, the third inequality in (2.3) implies that $a > 0$ and $-1 < p_3 < 1$. By noting this fact and the first two inequalities in (2.3), we see that

$$1 - p_1 + p_2 - p_3 > -p_3(1 - p_1 + p_2 - p_3) \quad \text{and} \quad 1 + p_1 + p_2 + p_3 > p_3(1 + p_1 + p_2 + p_3)$$

which yields

$$1 - p_3^2 + p_2 - p_1 p_3 > p_1 - p_2 p_3 \quad \text{and} \quad 1 - p_3^2 + p_2 - p_1 p_3 > -(p_1 - p_2 p_3).$$

Hence, $a > b$ and $a > -b$ and so it follows that $a^2 - b^2 > 0$.

Now, we show that $(a - c)(a(a + c) - 2b^2) > 0$ under the condition (2.3). To this end, we first show that $a > c$. In fact, it follows from the first two inequalities in (2.3) that

$$(1 + p_2^2) - (p_1 + p_3)^2 > 0$$

which yields

$$1 + p_2^2 + 2p_2 - p_1^2 - p_3^2 - 2p_1 p_3 > 0.$$ 

Hence,

$$1 - p_3^2 + p_2 - p_1 p_3 > p_1^2 - p_2^2 - p_2 + p_1 p_3$$
which is just $a > c$. Next we show that $a(a + c) - 2b^2 > 0$. From (2.3) we see that

$$(1 + p_1 + p_2 + p_3)(1 - p_1 + p_2 - p_3)(1 - p_3^2 - p_2 + p_1 p_3) > 0.$$ 

A straightforward calculation shows that the left-hand side of this inequality is the same as $a(a + c) - 2b^2$. Hence, $a(a + c) - 2b^2 > 0$ and so $(a - c)(a(a + c) - 2b^2) > 0$.

From the above discussion, we see that if (2.3) is satisfied, then $M_3 > 0$ and $V(y)$ is positive definite with

$$V(y) = \frac{(a-c)(a + ac - 2b^2)y_1^2}{a^2 - b^2} + \frac{1}{a} (by_1 + ay_2 + by_3)^2 + \frac{1}{a(a^2 - b^2)} [(ac - b^2)y_1 + (a^2 - b^2)y_3]^2$$

and hence the zero solution of (2.6) for $n = 3$ is asymptotically stable by Theorem B.

**Remark 2.1.** We will show in Section 3 that all roots of (2.1) lie within the unit circle on the complex plane if and only if $M_n > 0$ and $V(y)$ in (2.8) is positive definite.

Next, we adopt the notation from Tsai [20] and construct Liapunov functions for the system (1.1) by transforming (1.1) into (2.6). For each $x = (x_1, x_2, \ldots, x_n)^T \in \mathbb{R}^n$, replace the $j$-th column of $A$ by $x$ and denote the new matrix by $A_j(x)$. By striking out all columns and rows of $A_j(x)$ but columns $\nu_1, \nu_2, \ldots, \nu_k$ and rows $\nu_1, \nu_2, \ldots, \nu_k$ with $1 \leq \nu_1 < \nu_2, \ldots < \nu_k \leq n$ and $j \in \{\nu_1, \nu_2, \ldots, \nu_k\}$, we obtain a $k \times k$ submatrix of $A_j(x)$ and denote its determinant (a principle minor of order $k$) by $M_{\nu_1,\nu_2,\ldots,\nu_k}^{(j)}(x)$. Let $\sum M_{\nu_1,\nu_2,\ldots,\nu_k}^{(j)}(x)$ denote the sum of $M_{\nu_1,\nu_2,\ldots,\nu_k}^{(j)}(x)$, where $\nu_1, \nu_2, \ldots, \nu_k$ are taken from all possible combinations of numbers in the set $\{1, 2, \ldots, n\}$. For example, letting $n = 3$ and $j = 2$, we have

$$A_2(x) = \begin{vmatrix}
  a_{11} & x_1 & a_{13} \\
  a_{21} & x_2 & a_{23} \\
  a_{31} & x_3 & a_{33}
\end{vmatrix}$$

and

$$\sum M_{\nu_1}^{(2)}(x) = x_2, \quad \sum M_{\nu_1,\nu_2}^{(2)}(x) = \begin{vmatrix}
  a_{11} & x_1 \\
  a_{21} & x_2 \\
  a_{31} & x_3
\end{vmatrix} + \begin{vmatrix}
  x_2 & a_{23} \\
  x_3 & a_{33}
\end{vmatrix},$$
\[
\sum M_{1,2,3}^{(2)}(x) = \begin{vmatrix} a_{11} & x_1 & a_{13} \\ a_{21} & x_2 & a_{23} \\ a_{31} & x_3 & a_{33} \end{vmatrix}.
\]

Notice that \(\sum M^{(j)}_{\nu_1,\nu_2,\cdots,\nu_k}(x)\) plus the sum of all \(k\)-th order principle minors of \(A_j(x)\) without the \(x_j\) term is equal to \((-1)^k\) multiplied by the coefficient of \(\lambda^{n-k}\) in the characteristic polynomial \(|\lambda I - A_j(x)|\) for fixed \(x\) (see Lancaster and Tismenetsky [13], p.157). Moreover, \(\sum M^{(j)}_{\nu_1}(x) = x_j\) and \(\sum M^{(j)}_{1,2,\cdots,n}(x) = A_j(x)\).

The following lemma is due to Tsai [20] and was stated in terms of solutions to a system of linear ordinary differential equations. However, his proof is independent of the solutions. With the notation introduced above, we restate the lemma here in terms of matrix identities so that it can serve a more general purpose.

**Lemma 2.1** ([20]). Let \(A = (a_{ij})_{n \times n}\) be an \(n \times n\) matrix and \(x = (x_1, x_2, \cdots, x_n)^T \in \mathbb{R}^n\). Then for each \(j\) (\(j = 1, 2, \cdots, n\)), the following identities hold.

\[
\sum M^{(j)}_{\nu_1,\nu_2,\cdots,\nu_k}(Ax) = (-1)^k p_k x_j - \sum M^{(j)}_{\nu_1,\nu_2,\cdots,\nu_k,\nu_{k+1}}(x)
\]

for \(k = 1, 2, \cdots, n - 1\), and

\[
\sum M^{(j)}_{1,2,\cdots,n}(Ax) = (-1)^n p_n x_j.
\]

To further understand these equations, we verify (2.13) and (2.14) directly for \(n = 3\) and \(j = 2\). Replace \(x\) by \(Ax\) in the definition of \(M^{(2)}_{\nu_1,\nu_2,\cdots,\nu_k}(x)\) to obtain

\[
\sum M^{(2)}_{\nu_1}(Ax) = \sum_{k=1}^{3} a_{2k} x_k
\]

\[
= -p_1 x_2 - (a_{11} x_2 - a_{21} x_1) - (a_{33} x_2 - a_{23} x_3)
\]

\[
= -p_1 x_2 - \left( \begin{vmatrix} a_{11} & x_1 \\ a_{21} & x_2 \end{vmatrix} + \begin{vmatrix} x_2 & a_{23} \\ x_3 & a_{33} \end{vmatrix} \right)
\]

\[
= -p_1 x_2 - \sum M^{(2)}_{\nu_1,\nu_2}(x),
\]
\[
\sum M_{\nu_1, \nu_2, \nu_3}^{(2)}(Ax) = \begin{vmatrix} a_{11} \sum_{k=1}^{3} a_{1k}x_k & a_{13} \\ a_{21} \sum_{k=1}^{3} a_{2k}x_k & a_{23} \end{vmatrix} + \begin{vmatrix} \sum_{k=1}^{3} a_{2k}x_k & a_{23} \\ \sum_{k=1}^{3} a_{3k}x_k & a_{33} \end{vmatrix}
\]

\[
= \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} x_2 + \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} x_1 + \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} x_2 - \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} x_1 = p_2x_2 - \sum M_{\nu_1, \nu_2, \nu_3}^{(2)}(Ax),
\]

and

\[
\sum M_{\nu_1, \nu_2, \nu_3}^{(2)}(Ax) = \begin{vmatrix} a_{11} \sum_{k=1}^{3} a_{1k}x_k & a_{13} \\ a_{21} \sum_{k=1}^{3} a_{2k}x_k & a_{23} \\ a_{31} \sum_{k=1}^{3} a_{3k}x_k & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} x_1 + \begin{vmatrix} a_{11} & a_{13} & a_{13} \\ a_{21} & a_{23} & a_{23} \\ a_{31} & a_{33} & a_{33} \end{vmatrix} x_2 + \begin{vmatrix} a_{11} & a_{13} & a_{13} \\ a_{21} & a_{23} & a_{23} \\ a_{31} & a_{33} & a_{33} \end{vmatrix} x_3 = -p_3x_2.
\]

For fixed \(j\), we define a linear transformation \(L^{(j)}: R^n \rightarrow R^n\) by

\[
L^{(j)}(x) = (L_1^{(j)}(x), L_2^{(j)}(x), \cdots, L_n^{(j)}(x))^T
\]

(2.15)

where \(L_k^{(j)}(x) = \sum M_{\nu_1, \nu_2, \cdots, \nu_k}^{(j)}(x)\). Let \(y_k = L_k^{(j)}(x)\) or \(y = L^{(j)}(x)\). Notice that along a solution \(x = x(t)\) of (1.1), we have

\[
\sum M_{\nu_1, \nu_2, \cdots, \nu_k}^{(j)}(x(t+1)) = \sum M_{\nu_1, \nu_2, \cdots, \nu_k}^{(j)}(Ax).
\]

By virtue of (2.13) and (2.14), if \(x = x(t)\) is a solution of (1.1), then \(y = L^j(x(t))\) is also a solution of

\[
y_k(t+1) = (-1)^k p_ky_1(t) - y_{k+1}(t) \quad \text{for} \quad k = 1, 2, \cdots, n - 1,
\]

\[
y_n(t+1) = (-1)^n p_ny_1(t)
\]
which is the same as (2.6). We are now ready to state the following theorem.

**Theorem 2.2.** Let $V_j(x), x = (x_1, x_2, \cdots, x_n)^T$, be the quadratic form obtained by replacing the first row of $M_{n-1}$ by the vector

$$
\left( \sum_{\alpha - \beta = 0}^{\alpha - \beta = \pm k} L_\alpha^{(j)}(x)L_\beta^{(j)}(x), \cdots, (-1)^k \sum_{\alpha - \beta = \pm (n-1)} L_\alpha^{(j)}(x)L_\beta^{(j)}(x) \right).
$$

That is

$$
V_j(x) = \begin{vmatrix}
\sum_{\alpha - \beta = 0}^{\alpha - \beta = \pm 1} L_\alpha^{(j)}(x)L_\beta^{(j)}(x) & -\sum_{\alpha - \beta = 1}^{\alpha - \beta = \pm (n-1)} L_\alpha^{(j)}(x)L_\beta^{(j)}(x) & \cdots & (-1)^{n-1} \sum_{\alpha - \beta = \pm (n-1)} L_\alpha^{(j)}(x)L_\beta^{(j)}(x) \\
p_1 & p_2 + p_0 & \cdots & p_n + p_{2-n} \\
p_2 & p_3 + p_1 & \cdots & p_{1+n} + p_{3-n} \\
\cdots & \cdots & \cdots & \cdots \\
p_{n-1} & p_n + p_{n-2} & \cdots & p_{2(n-1)} + p_0
\end{vmatrix}
$$

$$
= \sum_{k=0}^{n-1} C_{k+1} \sum_{\alpha - \beta = \pm k} (-1)^k L_\alpha^{(j)}(x)L_\beta^{(j)}(x)
$$

where $C_k$ is the cofactor of the element in the first row and $k$-th column of $M_{n-1}$. Then

$$
V_j(Ax) - V_j(x) = -M_n x_j^2
$$

for $j = 1, 2, \cdots, n$.

Our next result gives a necessary and sufficient condition for the system (1.1) to be asymptotically stable

**Theorem 2.3.** If $V(x) = \sum_{j=1}^n V_j(x)$, then

$$
V(Ax) - V(x) = -M_n |x|^2
$$

and the zero solution of (1.1) is asymptotically stable if and only if $M_n > 0$ and $V(x)$ is positive definite.

**Proof.** Equation (2.17) follows from the definition of $V(x)$ and (2.16). If $M_n > 0$ and $V(x)$ is positive definite, then the zero solution of (1.1) is asymptotically stable by Theorem
B. Conversely, suppose that the zero solution of (1.1) is asymptotically stable; then \(|\lambda| < 1\) for each eigenvalue \(\lambda\) of \(A\). This in turn implies that \(M_n > 0\) (to be proved in Theorem 3.2 below). We now show that \(V(x)\) is positive definite. Let \(x_0 \in R^n\) with \(x_0 \neq 0\) and \(x(t) = x(t, t_0, x_0)\) be a solution of (1.1). Then

\[
V(x(t + 1)) = V(x(t)) - M_n|x(t)|^2
\]

for \(t \in N(t_0)\). Thus,

\[
V(x(t_0 + m + 1)) = V(x(t_0)) - M_n \sum_{j=0}^{m} |x(t_0 + j)|^2. \tag{2.18}
\]

Since the zero solution of (1.1) is asymptotically stable, we have \(x(t) \to 0\) as \(t \to \infty\). Let \(m \to \infty\) in (2.18) to get

\[
V(x_0) = V(x(t_0)) = M_n \sum_{j=0}^{\infty} |x(t_0 + j)|^2 > 0.
\]

This proves that \(V(x)\) is positive definite.

**Example 2.2.** Let \(j = 1\). Then Theorem 2.2 gives the following formulas. For \(n = 2\) and \(j = 1\),

\[
V_1(x) = (p_2 + 1) \left[ x_1^2 + \begin{vmatrix} x_1 & a_{12} \\ x_2 & a_{22} \end{vmatrix}^2 \right] + 2p_1 x_1 \begin{vmatrix} x_1 & a_{12} \\ x_2 & a_{22} \end{vmatrix}
\]

and

\[
V_1(Ax) - V_1(x) = -(1 - p_2)(1 + p_1 + p_2)(1 - p_1 + p_2)x_1^2
\]

For \(n = 3\) and \(j = 1\),

\[
V_1(x) = (1 - p_3^2 + p_2 - p_1 p_3) \left[ x_1^2 + \left( \begin{vmatrix} x_1 & a_{12} & a_{13} \\ x_2 & a_{22} & a_{23} \\ x_3 & a_{32} & a_{33} \end{vmatrix} \right)^2 + \begin{vmatrix} x_1 & a_{12} & a_{13} \\ x_2 & a_{22} & a_{23} \\ x_3 & a_{32} & a_{33} \end{vmatrix} \right]
\]

\[
+ 2(p_1 - p_2 p_3) \begin{vmatrix} x_1 & a_{12} & a_{13} \\ x_2 & a_{22} & a_{23} \\ x_3 & a_{32} & a_{33} \end{vmatrix} \left( \begin{vmatrix} x_1 & a_{12} \\ x_2 & a_{22} \end{vmatrix} + \begin{vmatrix} x_1 & a_{13} \\ x_3 & a_{33} \end{vmatrix} \right)
\]

\[
+ 2 \left[ p_1^2 - p_2^2 - (p_2 - p_1 p_3) \right] x_1 \begin{vmatrix} x_1 & a_{12} & a_{13} \\ x_2 & a_{22} & a_{23} \\ x_3 & a_{32} & a_{33} \end{vmatrix}
\]

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and
\[ V_1(Ax) - V_1(x) = - \sum_{i=0}^{3} p_i \sum_{i=0}^{3} (-1)^i p_i [(1 - p_i^2) - (p_2 - p_1 p_3)] x_1^2. \]

Note that this agrees with what we found in Example 2.1.

3 Stability Criteria for the Characteristic Polynomial and Evaluation of the Determinant \( M_n \)

The problem of root distribution of a polynomial with real (or complex) coefficients has been the subject of investigation of many mathematicians, physicists, and engineers in the last century. It can be traced back to the works of Routh-Hurwitz (see Lancaster and Tismenetsky [13]) and Schur-Cohn [3], [18]. In many problems in system theory, the clustering of roots in a certain region in the complex plane plays an important role. Such problems arise in stability, aperiodicity, transient response (relative stability), and many other related fields.

With the help of the Liapunov functions obtained in the last section, we shall derive a new determinantal criterion (necessary and sufficient condition) for all roots of the characteristic polynomial of \( A \) to lie within the unit circle on the complex plane. These necessary and sufficient conditions also help us to analyze properties of Liapunov functions.

We begin to evaluate the determinant \( M_n \) by introducing the “inners” of a matrix \( Q = (q_{ij})_{n \times n} \) (Jury [9] and Elaydi [5, p.180]). The inners of a matrix are the matrix itself and all the matrices obtained by striking out the first and last rows and the first and last columns.

For example, the inners of the matrix \( Q = (q_{ij})_{4 \times 4} \) are
\[
Q = \begin{pmatrix}
q_{11} & q_{12} & q_{13} & q_{14} \\
q_{21} & q_{22} & q_{23} & q_{24} \\
q_{31} & q_{32} & q_{33} & q_{34} \\
q_{41} & q_{42} & q_{43} & q_{44}
\end{pmatrix}, \quad
Q_2 = \begin{pmatrix}
q_{22} & q_{23} \\
q_{32} & q_{33}
\end{pmatrix}
\]
and the inners for the matrix \( Q = (q_{ij})_{5 \times 5} \) are

\[
Q = \begin{pmatrix}
q_{11} & q_{12} & q_{13} & q_{14} & q_{15} \\
q_{21} & q_{22} & q_{23} & q_{24} & q_{25} \\
q_{31} & q_{32} & q_{33} & q_{34} & q_{35} \\
q_{41} & q_{42} & q_{43} & q_{44} & q_{45} \\
q_{51} & q_{52} & q_{53} & q_{54} & q_{55}
\end{pmatrix}, \quad Q_3 = \begin{pmatrix}
q_{32} & q_{33} \\
q_{42} & q_{43} & q_{44} & q_{45} \\
\end{pmatrix}, \quad Q_1 = (q_{33}).
\]

**Definition 3.1.** A matrix \( Q = (q_{ij})_{n \times n} \) is said to be positive innerwise if the determinants of its inners are positive.

**Theorem C.** ([9]) The zeros of the characteristic polynomial (2.1) of \( A \) lie within the unit circle on the complex plane if and only if

(i) \( \sum_{j=0}^{n} p_j > 0 \),

(ii) \( \sum_{j=0}^{n} (-1)^j p_j > 0 \),

(iii) the \((n-1) \times (n-1)\) matrices \( B^+_{n-1} = X_{n-1} + Y_{n-1} \) and \( B^-_{n-1} = X_{n-1} - Y_{n-1} \) are positive innerwise, where

\[
X_{n-1} = \begin{pmatrix}
1 & 0 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
p_{n-3} & p_{n-4} & p_{n-5} & \cdots & 1 & 0 \\
p_{n-2} & p_{n-3} & p_{n-4} & \cdots & p_1 & 1
\end{pmatrix}
\]  \hspace{1cm} (3.1)

and

\[
Y_{n-1} = \begin{pmatrix}
0 & 0 & 0 & \cdots & 0 & p_n \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & p_n & \cdots & p_5 & p_4 \\
0 & p_n & p_{n-1} & \cdots & p_3 & p_2 \\
p_n & p_{n-1} & p_{n-2} & \cdots & p_3 & p_2
\end{pmatrix}.
\]  \hspace{1cm} (3.2)
Remark 3.1. Conditions (i)-(iii) are equivalent to those of Theorem A (Jury’s array). The cases $n = 2$ and $n = 3$ are given in (2.2) and (2.3).

To see a general procedure to evaluate $M_n$, we begin with the following example for $n = 4$.

$$
M_4 = \begin{vmatrix}
    p_0 & p_1 & p_2 & p_3 & p_4 \\
    p_1 & p_2 + p_0 & p_3 & p_4 & 0 \\
    p_2 & p_3 + p_1 & p_4 + p_0 & 0 & 0 \\
    p_3 & p_4 + p_2 & p_5 + p_1 & p_0 & 0 \\
    p_4 & p_5 + p_3 & p_6 + p_2 & p_1 & p_0
\end{vmatrix}
$$

where $p_0 = 1$ and $p_5 = p_6 = 0$. We perform the following operations.

Step 1. Add column $j$ ($j = 2, 3, 4, 5$) to column 1 and factor out $\sum_{j=0}^{4} p_j$ from the first column.

$$
M_4 = \sum_{j=0}^{4} p_j \begin{vmatrix}
    1 & p_1 & p_2 & p_3 & p_4 \\
    1 & p_2 + p_0 & p_3 & p_4 & 0 \\
    1 & p_3 + p_1 & p_4 + p_0 & 0 & 0 \\
    1 & p_4 + p_2 & p_5 + p_1 & p_0 & 0 \\
    1 & p_5 + p_3 & p_6 + p_2 & p_1 & p_0
\end{vmatrix}.
$$

Step 2. Add $(-1) \times$ row 1 to row $j$ ($j = 2, 3, 4, 5$).

$$
M_4 = \sum_{j=0}^{4} p_j \begin{vmatrix}
    1 & p_1 & p_2 & p_3 & p_4 \\
    0 & p_2 + p_0 - p_1 & p_3 & -p_2 & p_4 - p_3 & -p_4 \\
    0 & p_3 + p_1 - p_2 & p_4 + p_0 - p_2 & -p_3 & -p_4 \\
    0 & p_4 + p_2 - p_3 & p_5 + p_1 - p_2 & p_0 - p_3 & -p_4 \\
    0 & p_5 + p_3 - p_4 & p_6 + p_2 - p_3 & p_1 - p_4 & p_0 - p_4
\end{vmatrix}

= \sum_{j=0}^{4} p_j \begin{vmatrix}
    p_2 + p_0 - p_1 & p_3 & -p_2 & p_4 - p_3 & -p_4 \\
    p_3 + p_1 - p_2 & p_4 + p_0 - p_2 & -p_3 & -p_4 \\
    p_4 + p_2 - p_3 & p_5 + p_1 - p_2 & p_0 - p_3 & -p_4 \\
    p_5 + p_3 - p_4 & p_6 + p_2 - p_3 & p_1 - p_4 & p_0 - p_4
\end{vmatrix}.
$$
Step 3. Add column 3 to column 1 and factor out $\sum_{j=0}^{4}(-1)^jp_j$ from the first column.

\[ M_4 = \sum_{j=0}^{4} p_j \sum_{j=0}^{4} (-1)^j p_j \begin{bmatrix} 1 & p_3 & -p_2 & p_4 - p_3 & -p_4 \\ 0 & p_4 + p_0 - p_2 & -p_3 & -p_4 \\ 1 & p_5 + p_1 - p_2 & p_0 - p_3 & -p_4 \\ 0 & p_6 + p_2 - p_2 & p_1 - p_3 & p_0 - p_4 \end{bmatrix} \]

Step 4. Add $(-1) \times$ row 1 to row 3.

\[ M_4 = \sum_{j=0}^{4} p_j \sum_{j=0}^{4} (-1)^j p_j \begin{bmatrix} 1 & p_3 & -p_2 & p_4 - p_3 & -p_4 \\ 0 & p_4 + p_0 - p_2 & -p_3 & -p_4 \\ 0 & 0 & p_1 - p_3 & p_0 - p_4 \\ 0 & 0 & p_1 - p_3 & p_0 - p_4 \end{bmatrix} \]

Step 5. Add column 3 to column 1.

\[ M_4 = \sum_{j=0}^{4} p_j \sum_{j=0}^{4} (-1)^j p_j \begin{bmatrix} 1 - p_2 & -p_3 & -p_4 \\ p_1 - p_3 & 1 - p_4 & 0 \\ 1 - p_4 & p_1 - p_3 & 1 - p_4 \end{bmatrix} \]

Step 6. Add $(-1) \times$ row 1 to row 3.

\[ M_4 = \sum_{j=0}^{4} p_j \sum_{j=0}^{4} (-1)^j p_j \begin{bmatrix} 1 - p_2 & -p_3 & -p_4 \\ p_1 - p_3 & 1 - p_4 & 0 \\ p_1 - p_3 & p_0 & 0 \end{bmatrix} \begin{bmatrix} p_2 & p_3 & p_4 \\ p_0 & 0 \end{bmatrix} \]

\[ = \sum_{j=0}^{4} p_j \sum_{j=0}^{4} (-1)^j p_j \begin{bmatrix} 1 & 0 & 0 \\ p_1 & 1 & 0 \\ p_2 & p_1 & 1 \end{bmatrix} - \begin{bmatrix} p_2 & p_3 & p_4 \\ p_0 & 0 \end{bmatrix} \begin{bmatrix} p_2 & p_3 & p_4 \\ p_0 & 0 \end{bmatrix} \]

\[ = \sum_{j=0}^{4} p_j \sum_{j=0}^{4} (-1)^j p_j \begin{bmatrix} X_3 - Y_3 \end{bmatrix}. \]

The last equality holds since two $90^\circ$ degree rotations of a matrix do not change its determinant.
Remark 3.2. Clearly, a computer algorithm could be written to perform the above procedure.

Theorem 3.1. Let $M_n$ be given in (2.7). Then for $n \geq 2$,

$$M_n = \sum_{j=0}^{n} p_j \sum_{j=0}^{n} (-1)^j p_j |X_{n-1} - Y_{n-1}|$$

(3.3)

where $X_{n-1}$ and $Y_{n-1}$ are defined in (3.1) and (3.2), respectively.

Proof. Since the evaluation of $M_n$ is similar to that of $M_4$, we only outline the proof here. Without loss of generality, we may assume that $n$ is an even integer and proceed as follows.

1. Add column $j$ ($j = 2, 3, \cdots, n+1$) to column 1 and factor out $\sum_{j=0}^{n} p_j$ from the first column.
2. Add $(-1) \times$ row 1 to row $j$ ($j = 2, 3, \cdots, n+1$) and reduce the size of the matrix to $n \times n$.
3. Add column $j$ ($j = 3, 5, \cdots, n-1$) to column 1 and factor out $\sum_{j=0}^{n} (-1)^j p_j$ from the first column.
4. Add $(-1) \times$ row 1 to row $j$ ($j = 3, 5, \cdots, n-1$) and reduce the size the matrix to $(n-1) \times (n-1)$.
5. Add column $j$ ($j = 3, 5, \cdots, n-1$) to column 1. Add column $j$ ($j = 4, 6, \cdots, n-2$) column 2. Add column $j$ ($j = 5, 7, \cdots, n-1$) to column 3. Continue the operation to the last one adding column $(n-1)$ to column $(n-3)$.
6. Add $(-1) \times$ row $j$ to row $j+2$ ($j = (n-3), (n-4), \cdots, 2, 1$) starting with $j = n-3$.

Observing again the property that two 90° degree rotations of a matrix do not change its determinant, we see (3.3) holds. This completes the proof.

Example 3.1. It follows from Theorem 3.1 (also see Example 2.1) that

$$M_2 = (1 + p_1 + p_2)(1 - p_1 + p_2)(1 - p_2)$$

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and
\[
M_3 = \sum_{j=0}^{3} p_j \sum_{j=0}^{3} (-1)^j p_j \left( \begin{pmatrix} 1 & 0 \\ p_1 & 1 \end{pmatrix} - \begin{pmatrix} 0 & p_3 \\ p_3 & p_2 \end{pmatrix} \right) = \sum_{j=0}^{3} p_j \sum_{j=0}^{3} (-1)^j p_j \left[ 1 - p_3^2 - (p_2 - p_1 p_3) \right].
\]

**Theorem 3.2.** All roots of the characteristic polynomial of $A$ lie within the unit circle in the complex plane if and only if $M_n > 0$ and the matrix $B$ in (2.12) is positive definite.

**Proof.** Suppose that $|\lambda| < 1$ for each root $\lambda$ of the characteristic polynomial of $A$ in (2.1). Then the zero solution of (2.6) is asymptotically stable. This implies that each solution $y(t) = y(t, t_0, y_0)$ of (2.6) satisfies $y(t) \to 0$ as $t \to \infty$. It follows from Theorem C that
\[
\sum_{j=0}^{n} p_j > 0, \quad \sum_{j=0}^{n} (-1)^j p_j > 0, \quad \text{and} \quad |X_{n-1} - Y_{n-1}| > 0.
\]
This in turn implies that $M_n > 0$ by Theorem 3.1. We now show that $V(y)$ defined in (2.8) is positive definite. For each solution $y(t) = y(t, t_0, y_0)$ of (2.6), we have
\[
V(y(t + 1)) = V(y(t)) - M_n y_1^2(t)
\]
for all $t \in N(t_0)$, where $y(t) = (y_1(t), y_2(t), \cdots, y_n(t))^T$. Thus,
\[
V(y(t_0 + m + 1)) = V(y(t_0)) - M_n \sum_{j=0}^{m} y_1^2(t_0 + j)
\]
for all $m \geq 0$. Letting $m \to \infty$, we have
\[
V(y_0) = V(y(t_0)) = M_n \sum_{j=0}^{\infty} y_1^2(t_0 + j) \geq 0.
\]
Hence, $V(y) \geq 0$ for all $y \in R^n$. If there exists a $y_0 \neq 0$ such that $V(y_0) = 0$, by (3.4) $y_1(t) = 0$ for all $t \in N(t_0)$. However, this implies that $y(t) = y(t, t_0, y_0) = 0$ for all $t \in N(t_0)$ by (2.6). This yields $y_0 = 0$, which is a contradiction. Thus, $V(y)$ is positive definite or equivalently, the matrix $B$ in (2.12) is positive definite.
Conversely, suppose that $M_n > 0$ and $B$ is positive definite. Then $V(y)$ is positive definite. Since $V(y(t)) \leq V(y_0)$ for all $t \in N(t_0)$, solutions $y(t) = y(t, t_0, y_0)$ of (2.6) are bounded. By Theorem B, $y(t)$ approaches the set $M$ as $t \to \infty$, where $M$ is the maximal invariant subset of

$$E = \{ y \in R^n | V(Gy) - V(y) = -M_n y_1^2 = 0 \}$$

with respect to system (2.6). It is clear that $M = 0$. Thus, the zero solution of (2.6) is asymptotically stable. Therefore, all roots of the characteristic polynomial of $A$ lie within the unit circle. This completes the proof.

**Remark 3.3.** We can also state Theorem 3.2 as follows: All roots of the characteristic polynomial of $A$ lie within the unit circle if and only if $M_n > 0$ and the quadratic function $V(y)$ in (2.8) is positive definite.

**Example 3.2.** For $n = 2$ and $n = 3$, Theorem 3.2 becomes the following.

(i) All roots of the characteristic polynomial of $A$ with $n = 2$ lie within the unit circle if and only if $(1 + p_1 + p_2)(1 - p_1 + p_2)(1 - p_2) > 0$ and the matrix $\begin{pmatrix} p_2 + 1 & p_1 \\ p_1 & p_2 + 1 \end{pmatrix}$ is positive definite.

(ii) All roots of the characteristic polynomial of $A$ with $n = 3$ lie within the unit circle if and only if

$$\sum_{j=0}^{3} p_j \sum_{j=0}^{3} (-1)^j p_j \left[ 1 - p_3^2 - (p_2 - p_1p_3) \right] > 0$$

and the matrix

$$\begin{pmatrix} 1 - p_3^2 + p_2 - p_1p_3 & p_1 - p_2p_3 & p_1^2 - p_2^2 - (p_2 - p_1p_3) \\ p_1 - p_2p_3 & 1 - p_3^2 + p_2 - p_1p_3 & p_1 - p_2p_3 \\ p_1^2 - p_2^2 - (p_2 - p_1p_3) & p_1 - p_2p_3 & 1 - p_3^2 + p_2 - p_1p_3 \end{pmatrix}$$

is positive definite.

Another approach to determining stability properties of matrices can be based on the Lozinskii measure (see Coppel [4]) and the use of additive compound matrices. In this regard, we refer the reader to the paper of Muldowney [17] or the recent results of Li and Wang [15].
References


