CONTRACTION MAPPING AND STABILITY IN A DELAY-DIFFERENTIAL EQUATION

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ABSTRACT. In this paper we study the stability properties of a delay differential equation by means of contraction mappings. The paper is motivated by a number of difficulties encountered in the study of stability by means of Liapunov’s direct method. We notice that most of these difficulties vanish when applying fixed point theory. An asymptotic stability theorem with a necessary and sufficient condition is proved.

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1. INTRODUCTION

The purpose of this paper is to study the stability properties of the scalar delay equation

(1) \[ x'(t) = -a(t)x(t) + g(t, x_t) \]

by means of fixed point theory. Here \( a : \mathbb{R}^+ \to \mathbb{R} \) and \( g : \mathbb{R}^+ \times C \to \mathbb{R} \) are continuous with \( C \) being the Banach space of bounded continuous functions \( \phi : \mathbb{R} \to \mathbb{R} \) with the supremum norm \( \| \cdot \| \). If \( x : \mathbb{R} \to \mathbb{R} \) is a bounded continuous function and if \( t \geq 0 \) is a fixed number, then \( x_t \) denotes the restriction of \( x \) to the interval \( (-\infty, t] \) so that \( x_t \) is an element of \( C \) defined by \( x_t(s) = x(t+s) \) for \( s \in \mathbb{R}^- \).

When (1) is in a simple form, say

(2) \[ x'(t) = -a(t)x(t) + b(t)x(t - r(t)) \]

where \( b, r : \mathbb{R}^+ \to \mathbb{R} \) are continuous functions with

(3) \[ r(t) \geq 0, \ t - r(t) \to \infty \text{ as } t \to \infty \]

and

(4) \[ a(t) \geq \alpha, \ J|b(t)| \leq a(t), \ t \geq 0 \]

for some constants \( \alpha > 0, \ J > 1 \). One can apply Liapunov’s direct method to solve the problem. Taking the derivative of Liapunov function \( V(x) = x^2 \) along a solution \( x = x(t) \) of (2), we obtain

\[ V'(x) \leq -\mu x^2(t), \ \mu > 0 \]

whenever \( x^2(s) < Jx^2(t) \) for all \( s \in [t - r(t), t] \). It can now be argued that the zero solution of (2) is asymptotically stable (see [4, p.424]). Stability definitions may also be found in ([1], [4], [9]), for example.

Even in this simple example, it seems severe to ask \( a(t) \) be bounded away from zero and that \( |b(t)| \) is “bounded” by \( a(t) \) all the time. When \( g(t, \phi) \) is not a linear
functional, we may find many fundamental difficulties in the process of constructing a Liapunov function or functional (see [3]). However, we will show that most of these difficulties vanish when applying fixed point theory to (1). Our study here is restricted to the contraction mapping principle.

It is well-known that Liapunov’s direct method tends to require pointwise relation, but fixed point theory uses averaged conditions. This view will be well reflected in later two sections. For a comparison between the two methods, we refer the reader to ([2], [3]) and reference therein.

In Section 2, we prove a general stability theorem for (1) and compare the result with those obtained by Liapunov’s direct method. In Section 3, we give examples illustrating how to apply the main theorem to specific equations. It also contains some general results and remarks concerning the approach and should be viewed as an integral part of the paper.

2. STABILITY BY CONTRACTION MAPPING

Let \( R = (−∞, \infty), R^+ = [0, \infty), \) and \( R^- = (−∞, 0] \) respectively. For each \( γ > 0, \) define \( C(γ) = \{ \phi \in C : ∥\phi∥ ≤ γ \}. \) For a function \( ψ : R → R, \) we define \( ∥ψ∥_{[s,t]} = \sup\{|ψ(u)| : s ≤ u ≤ t \}. \) A continuous function \( x : (−∞, A) → R, A > 0 \) is called a solution of (1) through \((t_0, φ) ∈ R^+ × C\) if \( x_{t_0} = φ \) and \( x \) satisfies (1) on \([t_0, A]).\)

**Theorem 2.1.** Suppose that there exist positive constants \( α, L, \) and a continuous function \( b : R^+ → R^+ \) such that the following conditions hold:

(i) \( \lim\inf_{t→∞} \int_0^t a(s)ds > −∞. \)

(ii) \( \int_0^t e^{-\int_0^s a(u)du} b(s)ds ≤ α < 1 \) for all \( t ≥ 0. \)

(iii) \( |g(t, φ) − g(t, ψ)| ≤ b(t)∥φ − ψ∥ \) for all \( φ, ψ ∈ C(L), g(t, 0) = 0. \)

(iv) \( ∀ε > 0 \) and \( t_1 ≥ 0, \) there exists a \( t_2 > t_1 \) such that \( |t ≥ t_2, x(t) ∈ C(L)| \) imply

\( |g(t, x_t)| ≤ b(t) (ε + ∥x∥_{[t_1,t]}). \)

Then the zero solution of (1) is asymptotically stable if and only if

(v) \( \int_0^t a(s)ds → ∞ \) as \( t → ∞. \)

**Proof.** First, suppose that (v) holds. Let \( t_0 ≥ 0 \) and find \( δ_0 > 0 \) such that \( δ_0 K + αL ≤ L, \) where

\( K = \sup_{t ≥ t_0} \{e^{−\int_{t_0}^t a(s)ds}\}. \)

Let \( φ ∈ C(δ_0) \) be fixed and set

\( S = \{ x : R → R : x_{t_0} = φ, x_t ∈ C(L) \) for \( t ≥ t_0, x(t) → 0 \) as \( t → ∞ \}. \)

Then \( S \) is a complete metric space with metric \( ρ(x, y) = \sup_{t ≥ t_0} \{|x(t) − y(t)|\}. \)

Define \( P : S → S \) by

\( (Pφ)(t) = φ(t) \) for \( t ≤ t_0. \)
and
\[(P_x)(t) = \phi(t_0)e^{-\int_{t_0}^{t} a(s)ds} + \int_{t_0}^{t} e^{-\int_{s}^{t} a(u)du} f(s, x_s)ds \quad \text{for} \quad t \geq t_0.\]

Clearly, \((P_x) : R \rightarrow R\) is continuous with \((P_x)_{t_0} = \phi\) and
\[
|(P_x)(t)| \leq |\phi(t_0)|e^{-\int_{t}^{t_0} a(s)ds} + \int_{t_0}^{t} e^{-\int_{s}^{t} a(u)du} \|f(s, x_s)\|ds
\]
\[
\leq \delta_0K + \alpha L \leq L \quad \text{for} \quad t \geq t_0.
\]
Thus, \((P_x)_{t} \in C(L)\) for \(t \geq t_0\). We now show that \((P_x)(t) \rightarrow 0\) as \(t \rightarrow \infty\). Let \(x \in S\) and \(\varepsilon > 0\) be given. Since \(x(t) \rightarrow 0\) as \(t \rightarrow \infty\), there exists \(t_1 > t_0\) such that \(|x(t)| < \varepsilon\) for all \(t \geq t_1\). Since \(|x(t)| \leq L\) for all \(t \in R\), by (iv) there is \(t_2 > t_1\) such that \(t \geq t_2\) implies
\[
|g(t, x_t)| \leq b(t) \left(\varepsilon + \|x\|^{|t_1, t|}\right).
\]
For \(t \geq t_2\), we have
\[
\left|\int_{t_0}^{t} e^{-\int_{s}^{t} a(u)du} f(s, x_s)ds\right|
\]
\[
\leq \int_{t_0}^{t_2} e^{-\int_{s}^{t} a(u)du} \|f(s, x_s)\|ds + \int_{t_2}^{t} e^{-\int_{s}^{t} a(u)du} \|f(s, x_s)\|ds
\]
\[
\leq \int_{t_0}^{t_2} e^{-\int_{s}^{t} a(u)du} b(s)\|x_s\|ds + \int_{t_2}^{t} e^{-\int_{s}^{t} a(u)du} b(s) \left(\varepsilon + \|x\|^{|t_1, s|}\right)ds
\]
\[
\leq \alpha Le^{-\int_{t_0}^{t_2} a(s)ds} + 2\alpha \varepsilon.
\]
By (v), there exists \(t_3 > t_2\) such that
\[
\delta_0e^{-\int_{t_0}^{t_3} a(s)ds} + Le^{-\int_{t_2}^{t_3} a(s)ds} < \varepsilon.
\]
Thus, for \(t \geq t_3\), we have
\[
|(P_x)(t)| \leq \delta_0e^{-\int_{t_0}^{t} a(s)ds} + \alpha Le^{-\int_{t_2}^{t} a(s)ds} + 2\alpha \varepsilon < 3\varepsilon.
\]
This proves \((P_x)(t) \rightarrow 0\) as \(t \rightarrow \infty\), and hence \((P_x) \in S\).

To see that \(P\) is a contraction mapping, observe for \(t \geq t_0\)
\[
|(P_x)(t) - (P_y)(t)| \leq \int_{t_0}^{t} e^{-\int_{s}^{t} a(u)du} |g(s, x_s) - g(s, y_s)|ds
\]
\[
\leq \int_{t_0}^{t} e^{-\int_{s}^{t} a(u)du} b(s)\|x_s - y_s\|ds \leq \alpha \rho(x, y)
\]
or
\[
\rho(P_x, P_y) \leq \alpha \rho(x, y).
\]
By the Contraction Mapping Principle (see [5, p. 2]), \(P\) has a unique fixed point \(x\) in \(S\) which is a solution of (1) with \(\phi \in C(\delta_0)\) and \(x(t) = x(t, t_0, \phi) \rightarrow 0\) as \(t \rightarrow \infty\).

To obtain the asymptotic stability, we need to show that the zero solution of (1) is stable. Let \(\varepsilon > 0\) \((\varepsilon < L)\) be given. Choose \(\delta > 0\) \((\delta < \varepsilon)\) with \(\delta K + \alpha \varepsilon < \varepsilon\). If
$x(t) = x(t, t_0, \phi)$ is a solution of (1) with $\|\phi\| < \delta$, then

$$x(t) = \phi(t_0)e^{- \int_{t_0}^{t} a(s)ds} + \int_{t_0}^{t} e^{- \int_{t_0}^{u} a(u)du} g(s, x_s)ds.$$  

We claim that $|x(t)| < \varepsilon$ for all $t \geq t_0$. Notice that $|x(t)| < \varepsilon$. If there exists $t^* > t_0$ such that $|x(t^*)| = \varepsilon$ and $|x(s)| < \varepsilon$ for $t_0 \leq s < t^*$, then

$$|x(t^*)| < \delta e^{- \int_{t_0}^{t^*} a(s)ds} + \int_{t_0}^{t^*} e^{- \int_{t_0}^{u} a(u)du} b(s)^{x_s}ds \leq \delta K + \alpha \varepsilon < \varepsilon$$

which contradicts the definition of $t^*$. Thus, $|x(t)| < \varepsilon$ for all $t \geq t_0$ and the zero solution of (1) is stable. This shows that the zero solution of (1) is asymptotically stable if (v) holds.

Conversely, suppose (v) fails. Then by (i) there exists a sequence $\{t_n\}$, $t_n \to \infty$ as $n \to \infty$ such that

$$\lim_{n \to \infty} \int_{t_0}^{t_n} a(s)ds = \ell$$

for some $\ell \in \mathbb{R}$. We may also choose a positive constant $Q$ satisfying

$$-Q \leq \int_{0}^{t_n} a(s)ds \leq Q$$

for all $n = 1, 2, \cdots$. By (ii), we have

$$\int_{0}^{t_n} e^{- \int_{0}^{u} a(u)du} b(s)ds \leq \alpha.$$  

This yields

$$\int_{0}^{t_n} e^{\int_{0}^{u} a(u)du} b(s)ds \leq \alpha e^{\int_{0}^{t_n} a(u)du} \leq e^Q.$$  

The sequence $\{\int_{0}^{t_n} e^{\int_{0}^{u} a(u)du} b(s)ds\}$ is bounded so there exists a convergent subsequence. For brevity in notation, we may assume

$$\lim_{n \to \infty} \int_{0}^{t_n} e^{\int_{0}^{u} a(u)du} b(s)ds = \gamma$$

for some $\gamma \in \mathbb{R}^+$ and choose a positive integer $\bar{k}$ so large that

$$\int_{t_k}^{t_n} e^{\int_{0}^{u} a(u)du} b(s)ds < (1 - \alpha)/(2K^2)$$

for all $n \geq \bar{k}$.

By (i), $K$ in (5) is well defined. We now consider the solution $x(t) = x(t, t_\bar{k}, \phi)$ with $\phi(s) \equiv \delta_0$ for $s \leq t_\bar{k}$. Then $|x(t)| \leq L$ for all $t \geq t_\bar{k}$ and

$$|x(t)| \leq \delta_0 e^{- \int_{t_\bar{k}}^{t} a(s)ds} + \int_{t_\bar{k}}^{t} e^{- \int_{t_\bar{k}}^{u} a(u)du} b(s)ds \leq \delta_0 K + \alpha \|x_t\|.$$  

This implies that

$$|x(t)| \leq \frac{\delta_0 K}{1 - \alpha} =: \beta.$$
for all \( t \geq t_k \). On the other hand, for \( n \geq \bar{k} \), we also have

\[
|x(t_n)| \geq \delta_0 e^{-\int_{t_k}^{t_n} a(s)ds} - \int_{t_k}^{t_n} e^{-\int_{u}^{t_n} a(u)du} |b(s)||x_s|ds
\]

\[
\geq \delta_0 e^{-\int_{t_k}^{t_n} a(s)ds} - \beta e^{-\int_{t_k}^{t_n} a(s)ds} \int_{t_k}^{t_n} e^{\int_{u}^{t_n} a(u)du} b(s)ds
\]

\[
\geq e^{-\int_{t_k}^{t_n} a(s)ds} \left[ \delta_0 - \beta \int_{t_k}^{t_n} e^{\int_{u}^{t_n} a(u)du} b(s)ds \right]
\]

\[
\geq \frac{1}{2} \delta_0 e^{-\int_{t_k}^{t_n} a(s)ds} \geq \frac{1}{2} \delta_0 e^{-2Q}.
\]

This implies \( x(t) \not\to 0 \) as \( t \to \infty \). Thus, condition (v) is necessary for the asymptotic stability of the zero solution of (1). The proof is complete.

### 3. EXAMPLES AND REMARKS

In this section, we give several examples to illustrate how to apply Theorem 2.1 to some specific delay differential equations. Our emphasis will be on constructing a contraction mapping from the right-hand side of the equation. The examples are shown in simple forms for illustrative purpose, and they can be easily generalized.

**Example 3.1.** Consider the half-linear equation

\[
(6) \quad x'(t) = -a(t)x(t) + b(t)q(x(t-r(t)))
\]

where \( b, r : R^+ \to R \) and \( q : R \to R \) are continuous with

(i\*) \( \lim \inf_{t \to \infty} \int_0^t a(s)ds > -\infty \),

(ii\*) \( \sup_{t \geq 0} \int_0^t e^{-\int_0^u a(s)ds} |b(s)|ds < 1 \),

(iii\*) there exists an \( L > 0 \) so that if \( |x|, |y| \leq L \), then

\( |q(x) - q(y)| \leq |x - y| \) and \( q(0) = 0 \),

(iv\*) \( r(t) \geq 0, \ t - r(t) \to \infty \) as \( t \to \infty \).

Then the zero solution of (6) is asymptotically stable if and only if

(v\*) \( \int_0^t a(s)ds \to \infty \) as \( t \to \infty \).

Proof. One just applies Theorem 2.1 by replacing \( b(t) \) by \( |b(t)| \) in (ii)-(iv) and using the fading memory condition (iv\*) in (iv).

**Remark 3.1.** If \( q(x) \equiv x \), then (6) reduces to (2). It is clear that (4) implies (ii\*) and (v\*), and conditions here are significantly improved. We can see from (ii\*) that \( a(t) \) can be negative some of the time, \( a(t) \) and \( b(t) \) are related on average, and all functions involved can be unbounded.

**Remark 3.2.** It was noted in [8] that a fading memory condition such as (iv) or (iv\*) is required for the asymptotic stability of a general delay equation. This means that the equation representing a physical system should remember its past, but the memory should fade with time.
Example 3.2. Consider the Volterra equation

\[ x'(t) = -a(t)x(t) + \int_{-\infty}^{t} E(t, s, x(s))\,ds \tag{7} \]

where \( a : \mathbb{R}^+ \to \mathbb{R} \) and \( E : \Omega \times \mathbb{R} \to \mathbb{R}, \Omega = \{(t, s) \in \mathbb{R}^2 : t \geq s\} \) are continuous. Suppose there exist a constant \( L > 0 \) and a continuous function \( q : \Omega \to \mathbb{R}^+ \) such that

1. \( \liminf_{t \to -\infty} \int_{0}^{t} a(s)\,ds > -\infty \),
2. \( \sup_{t \geq 0} \int_{0}^{t} e^{-\int_{s}^{t} a(\tau)\,d\tau} \int_{-\infty}^{s} q(s, \tau)\,d\tau\,ds < 1 \),
3. if \( |x|, |y| \leq L \), then \( |E(t, s, x) - E(t, s, y)| \leq q(t, s)|x - y| \) and \( E(t, s, 0) = 0 \) for all \((t, s) \in \Omega, \)
4. \( \forall \varepsilon > 0 \) and \( t_1 \geq 0 \), there exists a \( t_2 > t_1 \) such that \( t \geq t_2 \) implies \( \int_{-\infty}^{t} q(t, s)\,ds \leq \varepsilon \int_{-\infty}^{t} q(t, s)\,ds \).

Then the zero solution of (7) is asymptotically stable if and only if

5. \( \int_{0}^{t} a(s)\,ds \to \infty \) as \( t \to \infty \).

Proof. We only need to verify that (iii) and (iv) hold. Indeed, letting \( g(t, \phi) = \int_{-\infty}^{0} E(t, t + s, \phi(s))\,ds \) and \( b(t) = \int_{0}^{t} q(t, s)\,ds \), we have

\[
|g(t, \phi) - g(t, \psi)| = \left| \int_{-\infty}^{0} E(t, t + s, \phi(s)) - \int_{-\infty}^{0} E(t, t + s, \psi(s))\,ds \right|
\leq \int_{-\infty}^{0} q(t, t + s)\,ds \|\phi - \psi\| = b(t) \|\phi - \psi\|
\]

for all \( \phi, \psi \in C(L) \). This shows (iii) holds. Next, let \( \varepsilon > 0 \) and \( t_1 \geq 0 \) be given. By (iv), there exists a \( t_2 > t_1 \) such that

\[
L \int_{-\infty}^{t_1} q(t, s)\,ds < \varepsilon \int_{-\infty}^{t} q(t, s)\,ds
\]

for all \( t \geq t_2 \). Let \( x : \mathbb{R} \to \mathbb{R} \) be continuous with \( x_t \in C(L) \). If \( t \geq t_2 \), then

\[
|g(t, x_t)| \leq \int_{-\infty}^{t_1} |E(t, s, x(s))|\,ds + \int_{t_1}^{t} |E(t, s, x(s))|\,ds
\leq \int_{-\infty}^{t_1} q(t, s)\,ds L + \int_{t_1}^{t} q(t, s)\,ds + |x(t)|\,ds
\leq \varepsilon \int_{-\infty}^{t} q(t, s)\,ds + \int_{t_1}^{t} q(t, s)\,ds \|x\|\,ds
\leq b(t) (\varepsilon + \|x\|\,ds) \bigg|_{t_1}^{t}.
\]

This implies that (iv) is satisfied, and by Theorem 2.1, the zero solution of (7) is asymptotically stable if and only if (v) holds.
Remark 3.3. When (7) is linear, say
\[ x'(t) = -a(t)x(t) + \int_{-\infty}^{t} C(t, s)x(s)ds, \]
a classical stability analysis by Liapunov’s method (see [7] and [10]) calls for point-wise conditions:
\[ \int_{0}^{\infty} a(t)dt = \infty, \]
(8) \[ -a(t) + K \int_{-\infty}^{t} |C(t, s)|ds \leq 0 \]
for all \( t \geq 0 \) and some constant \( K > 1 \), and
\[ \lim_{t \to \infty} \frac{1}{a(t)} \int_{-\infty}^{t} |C(t, s)|ds = 0 \]
for each \( t_1 \geq 0 \). This is automatically asking \( a(t) \geq 0 \) and \( \int_{0}^{t} |C(t, s)|ds \) be bounded by \( a(t) \) all the time. It is clear that (8) implies (ii).

Example 3.3. Finally, we consider the equation
\[ x'(t) = -\int_{0}^{t} C(t, s)x(s)ds \]
where \( C : \Omega \to R \) is continuous.

It is very important to point out that, concerning stability analysis, (9) should not be viewed as a special case of (7). To solve the problem, we introduce
\[ G(t, s) = \int_{t}^{\infty} C(u, s)du \]
and rewrite (9) as
\[ \frac{d}{dt} \left( x(t) - \int_{0}^{t} G(t, s)x(s)ds \right) = -Q(t)x(t). \]
If we define \( y(t) = x(t) - \int_{0}^{t} G(t, s)x(s)ds \), then \( x(t) \) can be represented by
\[ x(t) = y(t) - \int_{0}^{t} R(t, s)y(s)ds \]
where \( R(t, s) \) satisfies the resolvent equation
\[ R(t, s) = -G(t, s) + \int_{s}^{t} R(t, u)G(u, s)du. \]
We now rewrite (10) as
\[ y'(t) = -Q(t)y(t) + Q(t) \int_{0}^{t} R(t, s)y(s)ds. \]
It follows from Example 3.2 that if
\[ \hat{i} \lim_{t \to \infty} \int_{0}^{t} Q(s)ds > -\infty, \]
\[ \hat{i}i \sup_{t \geq 0} \int_{0}^{t} e^{-\int_{0}^{t} Q(u)du} |Q(s)| \left| \int_{0}^{s} R(s, \tau)d\tau \right| ds < 1, \]
(iii) there exists a constant $M > 0$ such that
\[ \int_0^t |R(t,s)| ds \leq M \text{ for } t \geq 0, \]

(iv) \( \forall \varepsilon > 0 \) and \( t_1 \geq 0 \), there exists a \( t_2 > t_1 \) such that \( t \geq t_2 \) implies \( \int_0^{t_1} |R(t,s)| ds \leq \varepsilon \),

then the zero solution of (9) is asymptotically stable if and only if

(v) \( \int_0^t Q(s) ds \to \infty \) as \( t \to \infty \).

Proof. By Example 3.2, the zero solution of (12) is asymptotically stable when (i-v) are satisfied. We only need to show that the asymptotic stability of (12) implies that of (9). Indeed, if \( x(t) = x(t, t_0, \phi) \) is a solution of (9), then \( y(t) = y(t, t_0, \psi) \) is a solution of (12) with \( \psi(t) = \phi(t) - \int_0^t G(t,s) \phi(s) ds \) for \( 0 \leq s \leq t_0 \). Then the asymptotic stability of (9) follows from an analysis of (11) by applying (iii) and (iv).

Remark 3.4. Condition (iii) can be stated in terms of \( G(t,s) \). For example, \( \sup_{t \geq 0} \int_0^t |G(t,s)| ds \leq \gamma < 1 \) implies \( \sup_{t \geq 0} \int_0^t |R(t,s)| ds < \infty \) by the resolvent equation. Another approach to determining stability properties of (9) can be based on the resolvent equation of (9) directly. In this regard, we refer the reader to ([6], [11]) and reference therein.

REFERENCES