ON REPRESENTATIONS OF QUANTUM GROUPS

$U_q(f(K, H))$

To Prof. Yingbo Zhang on the occasion of her 60th birthday

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Abstract: As further generalizations of $U_q(sl_2)$, an interesting class of algebras $U_q(f(K, H))$ was introduced and studied in [10]. In this paper, we further study these algebras by realizing them as Hyperbolic algebras. Such a realization yields a natural construction of interesting families of irreducible representations of $U_q(f(K, H))$ by using methods in non-commutative algebraic geometry([16]). We also investigate the relationship between $U_q(f(K, H))$ and the algebras $U_q(f(K))$ introduced in [11]. As an application, we prove that any finite dimensional weight representation of $U_q(f(K, H))$ is completely reducible. In addition, we study the Whittaker model for the center of $U_q(f(K, H))$. We prove that any Whittaker representation is irreducible if and only if it admits a central character. Thus, a complete classification of all irreducible Whittaker representations of $U_q(f(K, H))$ is obtained.

Keywords: Hyperbolic algebras, Spectral theory, Whittaker model, Quantum groups


0. Introduction

Several works have been focused on the study of generalizations (or deformations) of the quantized enveloping algebra $U_q(sl_2)$ ([17], [9], [11], [10], and [19]). Especially, a general class of algebras $U_q(f(K))$ (similar to $U_q(sl_2)$) was introduced in [11], and their finite dimensional representations were studied as well. And the representation theory of these algebras was further investigated in [19] from the perspectives of spectral theory and Whittaker model. In [10], as generalizations of the algebras $U_q(f(K))$, a new class of algebras $U_q(f(K, H))$ was introduced and studied. Note the Drinfeld quantum double of the positive part of the quantized enveloping algebra $U_q(sl_2)$ studied in [7] or equivalently the two-parameter quantum groups $U_{r,s}(sl_2)$ studied in [2] features as a special case of these algebras. The condition on the parameter Laurent polynomial $f(K, H) \in \mathbb{C}[K^{\pm 1}, H^{\pm 1}]$ for the existence of a Hopf algebra structure on $U_q(f(K, H))$ was determined, and finite dimensional irreducible representations were explicitly constructed as quotients of highest weight representations in [10]. This class of algebras provides a large family of quantum groups in the sense of Drinfeld ([4]).

It is not surprising that these algebras share a lot of similar properties with the two-parameter quantum groups $U_{r,s}(sl_2)$. So it might be useful to get more

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information on the representation theory of these algebras. In the first part of this paper, we will take a different approach to study the representations (not necessarily finite dimensional) of $U_q(f(K, H))$ from the point of view of spectral theory as developed in [16]. Namely, we will realize these algebras as Hyperbolic algebras, then apply the general results obtained in [16] on Hyperbolic algebras to construct interesting families of irreducible representations of $U_q(f(K, H))$. This approach yields the highest weight, the lowest weight and weight irreducible representations.

Note that there is also a close relationship between the representation theory of $U_q(f(K))$ and that of $U_q(f(K, H))$. We will explore this relationship following the idea in [7]. As an application, we obtain some nice results on the category of all weight representations of $U_q(f(K, H))$. In particular, we are able to show that the category of all weight representations of $U_q(f(K, H))$ is equivalent to the product of the category of weight representations of $U_q(f(K))$ with $\mathbb{C}^*$ as tensor categories. We have to admit that our results are motivated by the results in [7] for the Drinfeld double, and we follow their approach closely. Combined with results proved in [11], we are able to show that any finite dimensional weight representation of $U_q(f(K, H))$ is completely reducible.

Finally, we study the Whittaker model of the center for these algebras. We will prove that any Whittaker representation is irreducible if and only if it admits a central character. This result yields a complete classification of all irreducible Whittaker representations of $U_q(f(K, H))$.

Now let us mention a little bit about the organization of this paper. In Section 1, we recall the definition and basic facts about $U_q(f(K))$ and $U_q(f(K, H))$ from [11] and [10]. In Section 2, we recall some basic facts about spectral theory and Hyperbolic algebras from [16]. Then we realize $U_q(f(K, H))$ as Hyperbolic algebras and construct interesting families of irreducible weight representations. In Section 3, we study the relationship between $U_q(f(K))$ and $U_q(f(K, H))$ from the perspective of representation theory. In Section 4, we give a transparent description on the center of $U_q(f(K, H))$ and construct its Whittaker model. We study the Whittaker representations of $U(f(K, H))$, and a complete classification of irreducible Whittaker representations will be obtained. For the simplicity of computations, we will always assume that $f(K, H) = \frac{K^n - H^n}{q - q^{-1}}$ and $q$ is not a root of unity.

1. The algebras $U_q(f(K, H))$

Let $\mathbb{C}$ be the field of complex numbers and $0 \neq q \in \mathbb{C}$ such that $q^2 \neq 1$. The quantized enveloping algebra corresponding to the simple Lie algebra $sl_2$ is the associative $\mathbb{C}$–algebra generated by $K^{\pm 1}, E, F$ subject to the following relations:

$$KE = q^2EK, \quadKF = q^{-2} FK, \quad KK^{-1} = K^{-1}K = 1,$$

$$EF - FE = \frac{K - K^{-1}}{q - q^{-1}}.$$

This algebra is denoted by the standard notation $U_q(sl_2)$. It is well-known that $U_q(sl_2)$ is a Hopf algebra with the following Hopf algebra structure:

$$\Delta(E) = E \otimes 1 + K \otimes E, \quad \Delta(F) = F \otimes K^{-1} + 1 \otimes F;$$

$$\epsilon(E) = 0 = \epsilon(F), \quad \epsilon(K) = 1 = \epsilon(K^{-1});$$

$$s(E) = -K^{-1}E, \quad s(F) = -FK, \quad s(K) = K^{-1}.$$
As generalizations of $U_q(\mathfrak{sl}_2)$, a class of algebras $U_q(f(K))$ parameterized by Laurent polynomials $f(K) \in \mathbb{C}[K, K^{-1}]$ was later introduced in [11]. For the reader’s convenience, we recall their definition here.

**Definition 1.1.** (See [11]) For any Laurent polynomial $f(K) \in \mathbb{C}[K, K^{-1}], U_q(f(K))$ is the $\mathbb{C}$–algebra generated by $E, F, K^{\pm 1}$ subject to the following relations

$$KE = q^2EK, \quadKF = q^{-2}FK;$$

$$KK^{-1} = K^{-1}K = 1;$$

$$EF - FE = f(K).$$

The ring theoretic properties and finite dimensional representations were studied in detail in [11]. We state some of the results here without proof. First of all, for the Laurent polynomials $f(K) = a(K^m - K^{-m})$ where $a \in \mathbb{C}^*$ and $m \in \mathbb{N}$, the algebras $U_q(f(K))$ have a Hopf algebra structure. In particular, we have the following proposition quoted from [11].

**Proposition 1.1.** (Prop 3.3 in [11]) Assume $f(K)$ is a non-zero Laurent polynomial in $\mathbb{C}[K, K^{-1}]$. Then the non-commutative algebra $U_q(f(K))$ is a Hopf algebra such that $K, K^{-1}$ are group-like elements, and $E, F$ are skew primitive elements if and only if $f(K) = a(K^m - K^{-m})$ with $m = t - s$ and the following conditions are satisfied.

$$\Delta(K) = K \otimes K, \quad \Delta(K^{-1}) = K^{-1} \otimes K^{-1};$$

$$\Delta(E) = E^s \otimes E + E \otimes K^t, \quad \Delta(F) = K^{-t} \otimes F + F \otimes K^{-s};$$

$$\epsilon(K) = \epsilon(K^{-1}) = 1, \quad \epsilon(E) = \epsilon(F) = 0;$$

$$S(K) = K^{-1}, \quad S(K^{-1}) = K;$$

$$S(E) = -K^{-s}EK^{-t}, \quad S(F) = -K^tFK^s.\Box$$

For the case $f(K) = \frac{K^m - K^{-m}}{q - q^{-1}}$ for $m \in \mathbb{N}$ and $q$ is not a root of unity, the finite dimensional irreducible representations were proved to be highest weight and constructed explicitly in [11]. Furthermore, any finite dimensional representations are completely reducible as stated in the following theorem from [11].

**Theorem 1.1.** (Thm 4.17 in [11]) With the above assumption for $f(K)$ and $q$, any finite dimensional representation $V$ of $U_q(f(K))$ is completely reducible.\Box

**Remark 1.1.** The representation theory of $U_q(f(K))$ was studied further from the points of views of spectral theory and Whittaker model in [19], where more families of interesting irreducible representations were constructed.

As generalizations of the algebras $U_q(f(K))$, another general class of algebras parameterized by Laurent polynomials $f(K, H) \in \mathbb{C}[K^{\pm 1}, H^{\pm 1}]$ was also introduced and studied in [10]. First, let us recall the definition of $U_q(f(K, H))$ here:

**Definition 1.2.** (See [10]) Let $f(K, H) \in \mathbb{C}[K^{\pm 1}, H^{\pm 1}]$ be a Laurent polynomial, $U_q(f(K, H))$ is defined to be the $\mathbb{C}$–algebra generated by $E, F, K^{\pm 1}, H^{\pm 1}$ subject
to the following relations

\[ KE = q^2 EK, \quadKF = q^{-2} FK, \]
\[ HE = q^{-2} EH, \quad HF = q^2 FH, \]
\[ KK^{-1} = K^{-1} K = 1 = HH^{-1} = H^{-1} H, \quad KH = HK, \]
\[ EF - FE = f(K, H). \]

It is easy to see that the Drinfeld quantum double of the positive part of \( U_q(sl_2) \) or the two-parameter quantum group \( U_{r,s}(sl_2) \) features as a special case among these algebras. The condition on the parameter \( f(K, H) \) for the existence of a Hopf algebra structure on \( U_q(f(K, H)) \) was determined, and finite dimensional irreducible representations were constructed explicitly as quotients of highest weight representations in [10]. In addition, a counter example was also constructed to show that not all finite dimensional representations are completely reducible in [10]. So it would be interesting to know what kind of finite dimensional representations are completely reducible. We will address this question in Section 3.

2. Hyperbolic algebras and their representations

In this section, we realize \( U_q(f(K, H)) \) as Hyperbolic algebras and apply the methods in spectral theory as developed in [16] to construct irreducible weight representations of \( U_q(f(K, H)) \). For the reader’s convenience, we need to recall a little bit background about spectral theory and Hyperbolic algebras from [16].

2.1. Preliminaries on spectral theory. Spectral theory of abelian categories was first started by Gabriel in [5]. He defined the injective spectrum of any noetherian Grothendieck category. This spectrum consists of isomorphism classes of indecomposable injective objects. If \( R \) is a commutative noetherian ring, then the spectrum of the category of all \( R \)-modules is isomorphic to the prime spectrum \( \text{Spec}(R) \) of \( R \). And one can reconstruct any noetherian commutative scheme \( (X, O_X) \) using the spectrum of the category of quasi-coherent sheaves of modules on \( X \). The spectrum of any abelian category was later on defined by Rosenberg in [16]. This spectrum works for any abelian category. Via this spectrum, one can reconstruct any quasi-separated and quasi-compact commutative scheme \( (X, O_X) \) via the spectrum of the category of quasi-coherent sheaves of modules on \( X \).

Though spectral theory is more important for the purpose of non-commutative algebraic geometry, it has nice applications to representation theory. Spectrum has a natural analogue of the Zariski topology and its closed points are in a one to one correspondence with the irreducible objects of the category. To study irreducible representations, one can study the spectrum of the category of all representations, then single out closed points of the spectrum with respect to the associated topology. As an application of spectral theory to representation theory, points of the spectrum (hence representations) have been constructed for a large family of algebras called Hyperbolic algebras in [16]. It is a pure luck that a large family of ‘small’ algebras such as the first Weyl algebra \( A_1 \), \( U(sl_2) \) and their quantized versions and deformations are Hyperbolic algebras.

Now we are going to review some basic notions and facts about spectrum of any abelian category. First we review the definition of the spectrum of any abelian category, then we explain its applications in representation theory. We refer the reader to [16] for more details.
Let $C_X$ be an abelian category and $M, N \in C_X$ be any two objects; We say that $M \succ N$ if and only if $N$ is a sub-quotient of the direct sum of finite copies of $M$. It is easy to verify that $\succ$ is a pre-order. We say $M \approx N$ if and only if $M \succ N$ and $N \succ M$. It is obvious that $\approx$ is an equivalence. Let $Spec(X)$ be the family of all nonzero objects $M \in C_X$ such that for any non-zero sub-object $N$ of $M$, $N \succ M$.

**Definition 2.1.** (See [16]) The spectrum of any abelian category is defined to be:

$$\text{Spec}(X) = \text{Spec}(X)/\approx.$$  

Note that $\text{Spec}(X)$ has a natural analogue of Zariski topology. Its closed points are in a one to one correspondence with irreducible objects of $C_X$. To understand the application of spectral theory to representation theory, let us look at an example. Suppose $R$ is an associative algebra and $C_X$ is the category of all $R$−modules. Then closed points of $\text{Spec}(X)$ are in a one to one correspondence to irreducible $R$−modules. So the question of constructing irreducible $R$−modules is turned into a more natural question of studying closed points of the spectrum. The advantage is that we can study the spectrum via methods in non-commutative algebraic geometry. In addition, the left prime spectrum $\text{Spec}_l(R)$ of any ring $R$ is also defined in [16]. When $R$ is commutative, $\text{Spec}_l(R)$ is nothing but the prime spectrum $\text{Spec}(R)$. We have to mention that it is proven in [16] that $\text{Spec}_l(R)$ is isomorphic to $\text{Spec}(R - \text{mod})$ as topological spaces. So we will not distinguish the left prime spectrum $\text{Spec}_l(R)$ of a ring $R$ and the spectrum $\text{Spec}(R - \text{mod})$ of the category of all $R$−modules.

2.2. **Hyperbolic algebra** $R\{\xi, \theta\}$ and its spectrum. Hyperbolic algebras are studied by Rosenberg in [16] and by Bavula under the name of Generalized Weyl algebras in [1]. Hyperbolic algebra structure is very convenient for the construction of points of the spectrum. And a lot of interesting algebras such as the first Weyl algebra $A_1$, $U(sl_2)$ and their quantized versions have a Hyperbolic algebra structure. Points of the spectrum of the category of modules over Hyperbolic algebras are constructed in [16]. We review some basic facts about Hyperbolic algebras and two important construction theorems from [16].

Let $\theta$ be an automorphism of a commutative algebra $R$; and let $\xi$ be an element of $R$.

**Definition 2.2.** The Hyperbolic algebra $R\{\theta, \xi\}$ is defined to be the $R$−algebra generated by $x, y$ subject to the following relations:

$$xy = \xi, \quad yx = \theta^{-1}(\xi)$$

and

$$xa = \theta(a)x, \quad ya = \theta^{-1}(a)y$$

for any $a \in R$. And $R\{\theta, \xi\}$ is called a Hyperbolic algebra over $R$.

Let $C_X = C_{R\{\theta, \xi\}}$ be the category of modules over $R\{\theta, \xi\}$. We denote by $\text{Spec}(X)$ the spectrum of $C_X$. As we know that $\text{Spec}(X)$ is isomorphic to the left prime spectrum $\text{Spec}_l(R)$. Points of the left prime spectrum of Hyperbolic algebras are studied in [16], and in particular we have the following construction theorems from [16].

**Theorem 2.1.** (Thm 3.2.2. in [16])

1. Let $P \in \text{Spec}(R)$, and the orbit of $P$ under the action of the $\theta$ is infinite.
(a) If \( \theta^{-1}(\xi) \in P \) and \( \xi \in P \), then the left ideal
\[
P_{1,1} = P + R\{\theta, \xi\}x + R\{\theta, \xi\}y
\]
is a two-sided ideal from \( \text{Spec}_{\mathfrak{c}}(R\{\theta, \xi\}) \).

(b) If \( \theta^{-1}(\xi) \in P \), \( \theta^i(\xi) \notin P \) for \( 0 \leq i \leq n - 1 \), and \( \theta^n(\xi) \in P \), then the left ideal
\[
P_{1,n+1} = R\{\theta, \xi\}P + R\{\theta, \xi\}x + R\{\theta, \xi\}y^{n+1}
\]
belongs to \( \text{Spec}_{\mathfrak{c}}(R\{\theta, \xi\}) \).

(c) If \( \theta^i(\xi) \notin P \) for \( i \geq 0 \) and \( \theta^{-1}(\xi) \in P \), then
\[
P_{1,\infty} = R\{\theta, \xi\}P + R\{\theta, \xi\}x
\]
belongs to \( \text{Spec}_{\mathfrak{c}}(R\{\theta, \xi\}) \).

(d) If \( \xi \in P \) and \( \theta^{-i}(\xi) \notin P \) for all \( i \geq 1 \), then the left ideal
\[
P_{\infty,1} = R\{\theta, \xi\}P + R\{\theta, \xi\}y
\]
belongs to \( \text{Spec}_{\mathfrak{c}}(R\{\theta, \xi\}) \).

(2) If the ideal \( P \) in (b), (c) or (d) is maximal, then the corresponding left ideal of \( \text{Spec}_{\mathfrak{c}}(R\{\theta, \xi\}) \) is maximal.

(3) Every left ideal \( Q \in \text{Spec}_{\mathfrak{c}}(R\{\theta, \xi\}) \) such that \( \theta^\nu(\xi) \in Q \) for a \( \nu \in \mathbb{Z} \) is equivalent to one left ideal as defined above uniquely from a prime ideal \( P \in \text{Spec}(R) \). The latter means that if \( P \) and \( P' \) are two prime ideals of \( R \) and \( (\alpha, \beta) \) and \( (\nu, \mu) \) take values \( (1, \infty), (\infty, 1), (\infty, \infty) \) or \( (1, n) \), then \( P_{\alpha, \beta} \) is equivalent to \( P'_{\nu, \mu} \) if and only if \( \alpha = \nu, \beta = \mu \) and \( P = P' \).

\[ \square \]

**Theorem 2.2.** (Prop 3.2.3. in [16])

(1) Let \( P \in \text{Spec}(R) \) be a prime ideal of \( R \) such that \( \theta^i(\xi) \notin P \) for \( i \in \mathbb{Z} \) and \( \theta^i(P) - P \neq 0 \) for \( i \neq 0 \), then \( P_{\infty, \infty} = R\{\xi, \theta\}P \in \text{Spec}_{\mathfrak{c}}(R\{\xi, \theta\}) \).

(2) Moreover, if \( P \) is a left ideal of \( R\{\theta, \xi\} \) such that \( P \cap R = P \), then \( P = P_{\infty, \infty} \). In particular, if \( P \) is a maximal ideal, then \( P_{\infty, \infty} \) is a maximal left ideal.

(3) If a prime ideal \( P' \subset R \) is such that \( P_{\infty, \infty} = P'_{\infty, \infty} \), then \( P' = \theta^n(P) \) for some integer \( n \). Conversely, \( \theta^n(P)_{\infty, \infty} = P_{\infty, \infty} \) for any \( n \in \mathbb{Z} \).

\[ \square \]

2.3. **Realize \( U_q(f(K, H)) \) as Hyperbolic algebras.** Let \( R \) be the sub-algebra of \( U_q(f(K, H)) \) generated by \( EF, K^{\pm 1}, H^{\pm 1} \), then \( R \) is a commutative algebra. We define an algebra automorphism \( \theta: R \rightarrow R \) of \( R \) by setting
\[
\theta(EF) = EF + f(\theta(K), \theta(H)) ,
\theta(K^{\pm 1}) = q^{\pm 2}K^{\pm 1},
\theta(H^{\pm 1}) = q^{\pm 2}H^{\pm 1}.
\]

It is easy to see that \( \theta \) extends to an algebra automorphism of \( R \). Furthermore, we have the following lemma:
Lemma 2.1. The following identities hold:

\[ \begin{align*}
E(\mathcal{E}\mathcal{F}) &= \theta(\mathcal{E}\mathcal{F})E, \\
F(\mathcal{E}\mathcal{F}) &= \theta^{-1}(\mathcal{E}\mathcal{F})F, \\
\mathcal{E}\mathcal{K} &= \theta(\mathcal{K})E, \\
\mathcal{F}\mathcal{K} &= \theta^{-1}(\mathcal{K})F, \\
\mathcal{E}\mathcal{H} &= \theta(\mathcal{H})E, \\
\mathcal{F}\mathcal{H} &= \theta^{-1}(\mathcal{H})F.
\end{align*} \]

Proof: We only verify the first one and the rest of them can be checked similarly.

\[ \begin{align*}
E(\mathcal{E}\mathcal{F}) &= E(\mathcal{F}\mathcal{E} + f(\mathcal{K}, \mathcal{H})) \\
&= (\mathcal{E}\mathcal{F})E + Ef(\mathcal{K}, \mathcal{H}) \\
&= (\mathcal{E}\mathcal{F})E + f(\theta(\mathcal{K}), \theta(\mathcal{H}))E \\
&= (\mathcal{E} + f(\theta(\mathcal{K}), \theta(\mathcal{H})))E \\
&= \theta(\mathcal{E}\mathcal{F})E.
\end{align*} \]

So we are done.

From Lemma 2.1, we have the following result:

Proposition 2.1. \( U_q(f(\mathcal{K}, \mathcal{H})) = R\{\xi = \mathcal{E}\mathcal{F}, \theta\} \) is a Hyperbolic algebra with \( R \) and \( \theta \) defined as above.

It easy to see that we have the following corollary:

Corollary 2.1. (See also Prop. 2.5 in [10]) \( U_q(f(\mathcal{K}, \mathcal{H})) \) is noetherian domain of \( GK \)-dimension 4.

2.4. Families of irreducible weight representations of \( U_q(f(\mathcal{K}, \mathcal{H})) \). Now we can apply the above construction theorems to the case of \( U_q(f(\mathcal{K}, \mathcal{H})) \), and construct families of irreducible weight representations of \( U_q(f(\mathcal{K}, \mathcal{H})) \).

To proceed, we first state a lemma:

Lemma 2.2. Let \( M_{\alpha,\beta,\gamma} = (\xi - \alpha, \mathcal{K} - \beta, \mathcal{H} - \gamma) \subset R \) be a maximal ideal of \( R \), then \( \theta^n(M_{\alpha,\beta,\gamma}) \neq M_{\alpha,\beta,\gamma} \) for any \( n \geq 1 \). In particular, \( M_{\alpha,\beta,\gamma} \) has an infinite orbit under the action of \( \theta \).

Proof: We have

\[ \begin{align*}
\theta^n(\mathcal{K} - \beta) &= (q^{-2n}\mathcal{K} - \beta) \\
&= q^{-2n}(\mathcal{K} - q^{2n}\beta).
\end{align*} \]

Since \( q \) is not a root of unity, \( q^{2n} \neq 1 \) for any \( n \neq 0 \). So we have \( \theta^n(M_{\alpha,\beta,\gamma}) \neq M_{\alpha,\beta,\gamma} \) for any \( n \geq 1 \).

Now we construct all irreducible weight representations of \( U_q(f(\mathcal{K}, \mathcal{H})) \) for \( f(\mathcal{K}, \mathcal{H}) = a(\mathcal{K}^m - \mathcal{H}^m) \) with \( a \neq 0 \in \mathbb{C} \) and \( m \geq 1 \). Without loss of generality, we may assume that \( f(\mathcal{K}) = \frac{\mathcal{K}^m - \mathcal{H}^m}{q-1} \). We need another lemma:
Lemma 2.3.  
(1) For \( n \geq 0 \), we have the following:
\[
\theta^n(EF) = EF + \frac{1}{q - q^{-1}} \frac{q^{-2m}(1 - q^{-2nm})}{1 - q^{-2m}} K^m - \frac{q^{2m}(1 - q^{2nm})}{1 - q^{2m}} H^m.
\]
(2) For \( n \geq 1 \), we have the following:
\[
\theta^{-n}(EF) = EF - \frac{1}{q - q^{-1}} \frac{1 - q^{2nm}}{1 - q^{2m}} K^m - \frac{1 - q^{-2nm}}{1 - q^{-2m}} H^m.
\]
Proof: For \( n \geq 1 \), we have
\[
\theta^n(EF) = EF + \frac{1}{q - q^{-1}} ((q^{-2m} + \cdots + q^{-2nm}) K^m - (q^{2m} + \cdots + q^{2nm}) H^m).
\]
The second statement can be verified similarly. \( \square \)

Theorem 2.3.  
(1) If \( \alpha = \frac{\beta^{m-\gamma^m}}{q-q^{-1}} \), \((\beta/\gamma)^m = q^{2nm}\) for some \( n \geq 0 \), then \( \theta^n(\xi) \in M_{\alpha,\beta,\gamma} \) and \( \theta^{-1}(\xi) \in M_{\alpha,\beta,\gamma} \), thus \( U_q(f(K,H))/P_{1,n+1} \) is a finite dimensional irreducible representation of \( U_q(f(K,H)) \).
(2) If \( \alpha = \frac{\beta^{m-\gamma^m}}{q-q^{-1}} \) and \((\beta/\gamma)^m \neq q^{2nm}\) for all \( n \geq 0 \), then \( U_q(f(K,H))/P_{1,\infty} \) is an infinite dimensional irreducible representation of \( U_q(f(K,H)) \).
(3) If \( \alpha = 0 \) and \( 0 \neq \frac{1}{q-q^{-1}} \left( \frac{1}{q^{2m}} \frac{\beta^m}{1-q^{2m}} \gamma^m \right) \) for any \( n \geq 1 \), then \( U_q(f(K,H))/P_{\infty,1} \) is an infinite dimensional irreducible representation of \( U_q(f(K,H)) \).
Proof: Since \( \theta^{-1}(\xi) = \xi - \frac{K^m H^m}{q-q^{-1}} \), thus \( \theta^{-1}(\xi) \in M_{\alpha,\beta,\gamma} \) if and only if \( \alpha = \frac{\beta^{m-\gamma^m}}{q-q^{-1}} \). Now by the proof of Lemma 2.3, we have
\[
\theta^n(\xi) = \xi + \frac{1}{q - q^{-1}} ((q^{-2m} + \cdots + q^{-2nm}) K^m - (q^{2m} + \cdots + q^{2nm}) H^m).
\]
Hence \( \theta^n(\xi) \in M_{\alpha,\beta,\gamma} \) if and only if
\[
0 = \alpha + \frac{1}{q - q^{-1}} ((q^{-2m} + \cdots + q^{-2nm}) \beta^m - (q^{2m} + \cdots + q^{2nm}) \gamma^m)
\]
\[
= \alpha + \frac{1}{q - q^{-1}} (q^{-2m} \left( \frac{1 - q^{2nm}}{1 - q^{2m}} \right) \beta^m - \frac{q^{2m} (1 - q^{2nm})}{1 - q^{2m}} \gamma^m).
\]
Hence when \( \alpha = \frac{\beta^{m-\gamma^m}}{q-q^{-1}} \), \((\beta/\gamma)^m = q^{2nm}\) for some \( n \geq 0 \), we have
\[
\theta^n(\xi) \in M_{\alpha,\beta,\gamma}, \theta^{-1}(\xi) \in M_{\alpha,\beta,\gamma}.
\]
Thus by Theorem 2.1, \( U_q(f(K,H))/P_{1,n+1} \) is a finite dimensional irreducible representation of \( U_q(f(K,H)) \). So we have already proved the first statement, the rest of the statements can be similarly verified. \( \square \)

Remark 2.1. The representations we constructed in Theorem 2.3 exhaust all finite dimensional irreducible weight representations, the highest weight irreducible representations and the lowest weight irreducible representations of \( U_q(f(K,H)) \).
Representations by a non-zero scalar. As in [7], for each \( z \) algebra homomorphism \( \pi \)

\[
\operatorname{End}_{\mathbf{k}^{n}}\alpha \text{ and } \mathbf{k}^{n} = \mathbf{k}^{n}(\mathbf{c}, \mathbf{h}) \text{ for any } \mathbf{n} \geq 0 \text{ and } \alpha \neq \frac{1}{q - q^{-1}}(1 - q - q^{-1}) \alpha \text{ for any } \mathbf{n} \geq 1, \text{ then } \mathbf{U}_{q}(f(K, H))/\mathbf{P}_{\infty, \infty} \text{ is an infinite dimensional irreducible weight representation of } \mathbf{U}_{q}(f(K, H)).
\]

**Proof:** The proof is very similar to that of Theorem 2.3, we will omit it here. \( \Box \)

**Corollary 2.2.** The representations constructed in Theorem 2.3 and Theorem 2.4 exhaust all irreducible weight representations of \( \mathbf{U}_{q}(f(K, H)) \).

**Proof:** It follows directly from Theorems 2.1, 2.2, 2.3 and 2.4. \( \square \)

3. The relationship between \( \mathbf{U}_{q}(f(K)) \) and \( \mathbf{U}_{q}(f(K, H)) \)

We will denote by \( f(K, H) \) the polynomial \( \frac{K^{m} - H^{m}}{q - q^{-1}} \), and by \( f(K) \) the Laurent polynomial \( \frac{K^{m} - K^{-m}}{q - q^{-1}} \). We compare the algebras \( \mathbf{U}_{q}(f(K)) \) and \( \mathbf{U}_{q}(f(K, H)) \).

As a result, we will prove that any finite dimensional weight representation of \( \mathbf{U}_{q}(f(K, H)) \) is completely reducible.

First of all, it is easy to see that we have the following lemma generalizing the situation in [7]:

**Lemma 3.1.** The map which sends \( E \) to \( E, F \) to \( F, K^{\pm 1} \) to \( K^{\pm 1} \) and \( H^{\pm 1} \) to \( K^{\pm 1} \) extends to a unique surjective Hopf algebra homomorphism \( \pi: \mathbf{U}_{q}(f(K, H)) \rightarrow \mathbf{U}_{q}(f(K)) \).

**Proof:** The proof follows from the fact that the kernel of \( \pi \) is generated by \( K - H^{-1} \), which is a Hopf ideal of \( \mathbf{U}_{q}(f(K, H)) \). Note that both \( \mathbf{U}_{q}(f(K)) \) and \( \mathbf{U}_{q}(f(K, H)) \) are Hopf algebras under the assumption on \( f(K) \) and \( f(K, H) \). So we are done. \( \square \)

Our main goal in this section is to describe those representations \( M \) such that \( \operatorname{End}_{\mathbf{U}_{q}(f(K, H))}(M) = \mathbb{C} \). Since \( KH \) is in the center and invertible, it acts on these representations by a non-zero scalar. As in [7], for each \( z \in \mathbb{C}^{*} \), we define a \( \mathbb{C} \)-algebra homomorphism \( \pi_{z}: \mathbf{U}_{q}(f(K, H)) \rightarrow \mathbf{U}_{q}(f(K)) \) as follows:

\[
\pi_{z}(E) = z^{\frac{m}{2}} E, \quad \pi_{z}(F) = F; \quad \pi_{z}(K) = z^{\frac{1}{2}} K, \quad \pi_{z}(H) = z^{\frac{1}{2}} K^{-1}.
\]

It is easy to see that \( \pi_{z} \) is an algebra epimorphism with the kernel of \( \pi_{z} \) being a two-sided ideal generated by \( KH - z \). But they may not necessarily be a Hopf algebra homomorphism unless \( z = 1 \). Similar as in [7], we have the following lemma:

**Lemma 3.2.** For any representation \( M \) of \( \mathbf{U}_{q}(f(K, H)) \) such that \( \operatorname{End}_{\mathbf{U}_{q}(f(K, H))}(M) = \mathbb{C} \), there exists a unique \( z \in \mathbb{C}^{*} \) such that \( M \) is the pullback of a representation of \( \mathbf{U}_{q}(f(K)) \) through \( \pi_{z} \) as defined above. In particular, any such irreducible representation of \( \mathbf{U}_{q}(f(K, H)) \) is the pullback of an irreducible representation of \( \mathbf{U}_{q}(f(K)) \) through the algebra homomorphism \( \pi_{z} \) for some \( z \in \mathbb{C}^{*} \).
We use the notation in [7]. Let $M$ be a representation of $U_q(f(K))$, we denote by $M_z$ the representation of $U_q(f(K,H))$ obtained as the pullback of $M$ via $\pi_z$. Let $\epsilon_z$ be the one dimensional representation of $U_q(f(K,H))$ which is defined by mapping the generators of $U_q(f(K,H))$ as follows:

$$\epsilon_z(E) = \epsilon_z(F) = 0; \quad \epsilon_z(K) = z^{\frac{1}{2}}, \quad \epsilon_z(H) = z^{\frac{1}{2}}.$$ 

Then we have the following similar lemma as in [7]:

**Lemma 3.3.** Let $0 \neq z \in \mathbb{C}$, and $M$ be a representation of $U_q(f(K))$. Then $M_z \cong \epsilon_z \otimes M_1 \cong M_1 \otimes \epsilon_z$. In particular, if $0 \neq z' \in \mathbb{C}$ and $N$ is another representation of $U_q(f(K))$, then we have

$$M_z \otimes N_{z'} \cong (M \otimes N)_{zz'}.$$ 

**Proof:** The proof is straightforward.

Corollary 3.1. Any finite dimensional weight representation of $U_q(f(K,H))$ is completely reducible.

**Proof:** This follows from the above theorem and the fact that any finite dimensional representation of $U_q(f(K))$ is completely reducible (which is proved in [11]).

**Corollary 3.2.** The tensor product of any two finite dimensional weight representations is completely reducible.

**Remark 3.1.** After the first draft of this paper was written, we have been kindly informed by J. Hartwig that the complete reducibility of finite dimensional weight representations is also proved in his preprint [6] in a more general setting of Ambiskew polynomial rings via a different approach.

**Remark 3.2.** It might be interesting to study the decomposition of the product of two finite dimensional irreducible weight representations.

**Remark 3.3.** When $m = 1$, the above results are obtained in [7] for the Drinfeld double of the positive part of $U_q(sl_2)$, and equivalently for the two-parameter quantum groups $U_{r,s}(sl_2)$ in [2].
4. The Whittaker model for the center $Z(U_q(f(K, H)))$

Let $\mathfrak{g}$ be a finite dimensional complex semisimple Lie algebra and $U(\mathfrak{g})$ be its universal enveloping algebra. The Whittaker model for the center of $U(\mathfrak{g})$ was first studied by Kostant in [12]. The Whittaker model for the center $Z(U(\mathfrak{g}))$ is defined by a non-singular character of the nilpotent Lie subalgebra $\mathfrak{n}^+$ of $\mathfrak{g}$. Using the Whittaker model, Kostant studied the structure of Whittaker modules of $U(\mathfrak{g})$ and a lot of important results about Whittaker modules were obtained in [12]. Later on, Kostant’s idea was further generalized by Lynch in [13] and by Macdowell in [14] to the case of singular characters of $\mathfrak{n}^+$ and similar results hold in these generalizations.

The obstacle of generalizing the Whittaker model to the quantized enveloping algebra $U_q(\mathfrak{g})$ with $\mathfrak{g}$ of higher ranks is that there is no non-singular character existing for the positive part $(U_q(\mathfrak{g}))^{\geq 0}$ of $U_q(\mathfrak{g})$ because of the quantum Serre relations. In order to overcome this difficulty, it was Sevostyanov who first realized to use the topological version $U_h(\mathfrak{g})$ over $\mathbb{C}[[h]]$ of quantum groups. Using a family of Coxeter realizations $U_s^h(\mathfrak{g})$ of the quantum group $U_h(\mathfrak{g})$ indexed by the Coxeter elements $s$, he was able to prove Kostant’s results for $U_h(\mathfrak{g})$ in [18]. While in the simplest case of $\mathfrak{g} = sl_2$, the quantum Serre relations are trivial, thus a direct approach should still work and this possibility has been worked out recently in [15].

In addition, it is reasonable to wonder whether the Whittaker model exists for most of the deformations of $U_q(sl_2)$, since they are so close to $U_q(sl_2)$ and share a lot of properties with $U_q(sl_2)$. In this section, we show that there is such a Whittaker model for the center of $U_q(f(K, H))$, and will study the Whittaker modules for $U_q(f(K, H))$. We obtain parallel results as in [12] and [15]. In deed, the approach used here is more or less the same as the ones in [12] and [15]. For the reader’s convenience, we will work out all the details here. And we will use the term Whittaker modules instead of Whittaker representations for convenience.

4.1. The center $Z(U_q(f(K, H)))$ of $U_q(f(K, H))$. In this subsection, we first give a complete description of the center of $U_q(f(K, H))$. The center $Z(U_q(f(K, H)))$ was also studied in [10] as well. However, the description of $Z(U_q(f(K, H)))$ is incomplete there.

As mentioned from the very beginning, we always assume $f(K, H) = \frac{K^m - H^m}{q^m - q^{-m}}$ and $q$ is not a root of unity. We define a Casimir element of $U_q(f(H, K))$ by setting:

$$\Omega = FE + \frac{q^{2m}K^m + H^m}{(q^{2m} - 1)(q - q^{-1})}.$$ 

Then we have the following:

**Proposition 4.1.**

$$\Omega = \begin{array}{ccl}
FE + \frac{q^{2m}K^m + H^m}{(q^{2m} - 1)(q - q^{-1})} \\
= EF + \frac{K^m + q^{2m}H^m}{(q^{2m} - 1)(q - q^{-1})}.
\end{array}$$
Proof: Since $EF = FE + \frac{K^m - H^m}{q - q^{-1}}$, we have

$$
\Omega = FE + \frac{q^{2m}K^m + H^m}{(q^{2m} - 1)(q - q^{-1})} = FE - \frac{K^m - H^m}{q - q^{-1}} + \frac{q^{2m}K^m + H^m}{(q^{2m} - 1)(q - q^{-1})} = EF + \frac{K^m + q^{2m}H^m}{(q^{2m} - 1)(q - q^{-1})}.
$$

So we are done.

In addition, we have the following lemma:

**Lemma 4.1.** $\Omega$ is in the center of $U_q(f(K, H))$.

**Proof:** It suffices to show that $\Omega E = E\Omega, \Omega F = F\Omega, \Omega K = K\Omega, \Omega H = H\Omega$. We will only verify that $\Omega E = E\Omega$ and the rest of them are similar.

$$
\Omega E = (FE + \frac{q^{2m}K^m + H^m}{(q^{2m} - 1)(q - q^{-1})})E = (EF - \frac{K^m - H^m}{q - q^{-1}} + \frac{q^{2m}K^m + K^m}{(q^{2m} - 1)(q - q^{-1})})E = E(FE) + \frac{K^m + q^{2m}H^m}{(q^{2m} - 1)(q - q^{-1})}E = E(FE) + \frac{q^{2m}K^m + H^m}{(q^{2m} - 1)(q - q^{-1})}E = E\Omega.
$$

So we are done with the proof.

In particular, we have the following description of the center $Z(U_q(f(K)))$ of $U_q(f(K, H))$, which refines a result in [10].

**Proposition 4.2.** (See also [10]) $Z(U_q(f(K, H)))$ is the subalgebra of $U_q(f(K, H))$ generated by $\Omega, (KH)_{\pm 1}$. In particular, $Z(U_q(f(K, H)))$ is isomorphic to the localization $(\mathbb{C}[\Omega, KH])_{(\Omega, KH)}$ of the polynomial ring in two variables $\Omega, KH$.

**Proof:** By Lemma 3.1., we have $\Omega, (KH)_{\pm 1} \in Z(U_q(f(K, H)))$. Thus the subalgebra $\mathbb{C}[\Omega, (KH)_{\pm 1}]$ generated by $\Omega, (KH)_{\pm 1}$ is contained in $Z(U_q(f(K, KH)))$. So it suffices to prove the other inclusion $Z(U_q(f(K, H))) \subseteq \mathbb{C}[\Omega, (KH)_{\pm 1}]$. Note that $U_q(f(K, H)) = \bigoplus_{n \in \mathbb{Z}_{\geq 0}} U_q(f(K, H))_n$ where $U_q(f(K, H))_n$ is the $\mathbb{C}$-span of elements $\{u \in U_q(f(K, H)) \mid Ku = q^{2n}uK, Hu = q^{-2n}uH\}$. Suppose $x \in Z(U_q(f(K, H)))$, then $xK = Kx, xH = Hx$. Thus $x \in U_q(f(K))_0$ which is generated by $FE, K_{\pm 1}, H_{\pm 1}$. By the definition of $\Omega$, we know that $U_q(f(K))_0$ is also generated by $\Omega, K_{\pm 1}, H_{\pm 1}$. Hence $x = \sum f_{i,j}(\Omega)K^iH^j$ where $f_{i,j}(\Omega)$ are polynomials in $\Omega$. Therefore

$$
xE = \sum f_{i,j}(\Omega)K^iH^jE = \sum f_{i,j}(\Omega)q^{2i-2j}EK^iH^j = Ex,
$$

which forces $i = j$. So $x \in \mathbb{C}[\Omega, (KH)_{\pm 1}]$ as desired. So we have proved that $Z(U_q(f(K, H))) = \mathbb{C}[\Omega, (KH)_{\pm 1}]$. □
4.2. The Whittaker model for $Z(U_q(f(K, H)))$. Now we construct the Whittaker model for $Z(U_q(f(K, H)))$ following the lines in [12] and [15].

We first fix some notations. We denote by $U_q(E)$ the subalgebra of $U_q(f(K, H))$ generated by $E$, by $U_q(F, K^{\pm 1}, H^{\pm 1})$ the subalgebra of $U_q(f(K, H))$ generated by $F, K^{\pm 1}, H^{\pm 1}$. A non-singular character of the algebra $U_q(E)$ is defined as follows:

**Definition 4.1.** An algebra homomorphism $\eta: U_q(E) \to \mathbb{C}$ is called a non-singular character of $U_q(E)$ if $\eta(E) \neq 0$.

From now on, we fix such a non-singular character of $U_q(E)$ and denote it by $\eta$. Following [12], we define the concepts of a Whittaker vector and a Whittaker module corresponding to the fixed non-singular character $\eta$.

**Definition 4.2.** Let $V$ be a $U_q(f(K, H))$-module, a vector $0 \neq v \in V$ is called a Whittaker vector of type $\eta$ if $E$ acts on $v$ through the non-singular character $\eta$, i.e., $Ev = \eta(E)v$. If $V = U_q(f(K, H))v$, then we call $V$ a Whittaker module of type $\eta$ and $v$ is called a cyclic Whittaker vector of type $\eta$.

The following decomposition of $U_q(f(K, H))$ is obvious:

**Proposition 4.3.** $U_q(f(K, H))$ is isomorphic to $U_q(F, K^{\pm 1}, H^{\pm 1}) \otimes_{\mathbb{C}} U_q(E)$ as vector spaces and $U_q(f(K, H))$ is a free module over the subalgebra $U_q(E)$.

Let us denote the kernel of $\eta: U_q(E) \to \mathbb{C}$ by $U_q,\eta(E)$, and we have the following decompositions of $U_q(E)$ and $U_q(f(K, H))$.

**Proposition 4.4.** We have $U_q(E) = \mathbb{C} \oplus U_q,\eta(E)$. In addition,

\[
U_q(f(K, H)) \cong U_q(F, K^{\pm 1}, H^{\pm 1}) \oplus U_q(f(K, H))U_q,\eta(E).
\]

**Proof:** It is obvious that $U_q(E) = \mathbb{C} \oplus U_q,\eta(E)$. And we have

\[
U_q(f(K, H)) = U_q(F, K^{\pm 1}, H^{\pm 1}) \otimes (\mathbb{C} \oplus U_q,\eta(E)),
\]

thus

\[
U_q(f(K, H)) \cong U_q(F, K^{\pm 1}, H^{\pm 1}) \oplus U_q(f(K, H))U_q,\eta(E).
\]

So we are done.

Now we define a projection:

\[
\pi: U_q(f(K, H)) \to U_q(F, K^{\pm 1}, H^{\pm 1})
\]

from $U_q(f(K, H))$ onto $U_q(F, K^{\pm 1}, H^{\pm 1})$ by taking the $U_q(F, K^{\pm 1}, H^{\pm 1})$-component of any $u \in U_q(f(K, H))$. We denote the image $\pi(u)$ of $u \in U_q(f(K, H))$ by $u^\eta$ for short.

**Lemma 4.2.** If $v \in Z(U_q(f(K, H)))$ and $u \in U_q(f(K, H))$, then we have $u^\eta v^\eta = (uv)^\eta$.

**Proof:** Let $v \in Z(U_q(f(K, H)))$, $u \in U_q(f(K, H))$, then we have

\[
vw - u^\eta v^\eta = (u - u^\eta)v + u^\eta(v - v^\eta)
= v(u - u^\eta) + u^\eta(v - v^\eta),
\]

which is in $U_q(f(K, H))U_q,\eta(E)$. Hence $(uv)^\eta = u^\eta v^\eta$. \qed
Proposition 4.5. The map
\[ \pi: Z(U_q(f(K, H)) \rightarrow U_q(F, K^{\pm 1}, H^{\pm 1}) \]
is an algebra isomorphism of \( Z(U_q(f(K, H))) \) onto its image \( W(F, K^{\pm 1}, H^{\pm 1}) \) in \( U_q(F, K^{\pm 1}, H^{\pm 1}) \).

Proof: It follows from that Lemma 4.2. that \( \pi \) is a homomorphism of algebras. Note that \( \pi(\Omega) = \eta(E)F + \frac{q^{2n} - \eta^{-1}(q)(q-q^{-1})}{[n][q][q^{-1}]} \) with \( \eta(E) \neq 0 \). Furthermore, \( \pi(KH) = KH \). Since \( Z(U_q(f(K, H))) = \mathbb{C}[\Omega, (KH)^{\pm 1}] \), so \( \pi \) is an injection. Thus \( \pi \) is an algebra isomorphism from \( Z(U_q(f(K, H))) \) onto its image \( W(F, K^{\pm 1}, H^{\pm 1}) \) in \( U_q(F, K^{\pm 1}, H^{\pm 1}) \). \( \square \)

Lemma 4.3. If \( u^n = u \), then we have
\[ u^n v^n = (uv)^n \]
for any \( v \in U_q(f(K, H)) \).

Proof: We have
\[
uv - u^n v^n = (u - u^0)v + u^0(v - v^0) = u^0(v - v^0),
\]
which is in \( U_q(f(K, H))U_{q, \eta}(E) \). So we have
\[
u^n v^n = (uv)^n \]
for any \( v \in U_q(f(K, H)) \). \( \square \)

Let \( \tilde{A} \) be the subspace of \( U_q(f(K, H)) \) spanned by \( K^{\pm i}, H^{\pm 1} \) where \( i \in \mathbb{Z}_{\geq 0} \). Then \( \tilde{A} \) is a graded vector space with
\[
\tilde{A}[n] = \mathbb{C}K^{\pm n} \oplus \mathbb{C}H^{\pm n}
\]
for \( n \geq 1 \), and
\[
A[0] = \mathbb{C},
\]
and
\[
\tilde{A}[n] = 0
\]
for \( n \leq -1 \).

As in [12] and [15], we define a filtration of \( U_q(F, K^{\pm 1}, H^{\pm 1}) \) as follows:
\[
U_q(F, K^{\pm 1}, H^{\pm 1})[n] = \oplus_{i+m+j-k \leq n} U_q(F, K^{\pm 1}, H^{\pm 1})_{i,j,k}
\]
with \( U_q(F, K^{\pm 1}, H^{\pm 1})_{i,j,k} \) being the vector space spanned by \( F^i K^j H^k \). Since \( W(F, K^{\pm 1}, H^{\pm 1}) \) is a subalgebra of \( U_q(F, K^{\pm 1}, H^{\pm 1}) \), it inherits a filtration from \( U_q(F, K^{\pm 1}, H^{\pm 1}) \). We denote by
\[
W(F, K^{\pm 1}, H^{\pm 1})[p] = \mathbb{C}[(KH)^{\pm 1}] - \text{span}\{1, \Omega^n, \cdots, (\Omega^n)^p\}
\]
for \( q \geq 0 \). It is easy to see that
\[
W(F, K^{\pm 1}, H^{\pm 1})[p] \subset W(F, K^{\pm 1}, H^{\pm 1})[p+1],
\]
and
\[
W(F, K^{\pm 1}, H^{\pm 1}) = \sum_{p \geq 0} W(F, K^{\pm 1}, H^{\pm 1})[p].
\]
We have to mention that \( W(F, K^{\pm 1}, H^{\pm 1})[p] \) give a filtration of \( W(F, K^{\pm 1}, H^{\pm 1}) \) which is compatible with the filtration of \( U_q(F, K^{\pm 1}, H^{\pm 1}) \). In particular,
\[
W(F, K^{\pm 1}, H^{\pm 1})[p] = W(F, K^{\pm 1}, H^{\pm 1}) \cap U_q(F, K^{\pm 1}, H^{\pm 1})[p]
\]
for $p \geq 0$.

From the definition of $\Omega$, we have the following description of $\eta(\Omega)$, the image of $\Omega$ under $\eta$:

**Lemma 4.4.**

$$\eta(\Omega) = \eta(E)F + \frac{q^{2m}K^m + H^m}{(q^{2m} - 1)(q - q^{-1})}.$$  

\[\square\]

Furthermore, we have the same useful lemma about $\Omega^n$ as in [15] following from direct computations:

**Lemma 4.5.** Let $x = \Omega^n$ and $\xi = \eta(E)$. Then for $n \in \mathbb{Z}_{\geq 0}$. We have

$$x^n = \xi^n F^n + s_n,$$

where $s_n \in U_q(F, K^{\pm 1}, H^{\pm 1})_{[n]}$ and is a sum of terms of the forms $F^i K^j H^k \in U_q(F, K^{\pm 1}, H^{\pm 1})_{[n]}$ with $i < n$. In particular, $s_n$ are independent of $F^n$, whenever it is nonzero.

\[\square\]

Now we have the following decomposition of $U_q(F, K^{\pm 1}, H^{\pm 1})$.

**Theorem 4.1.** $U_q(F, K^{\pm 1}, H^{\pm 1})$ is free (as a right module) over $W(F, K^{\pm 1}, H^{\pm 1})$. And the multiplication induces an isomorphism

$$\Phi: \tilde{A} \otimes W(F, K^{\pm 1}, H^{\pm 1}) \longrightarrow U_q(F, K^{\pm 1}, H^{\pm 1})$$

as right $W(F, K^{\pm 1}, H^{\pm 1})$–modules. In particular, we have the following

$$\bigoplus_{p+i+m=nm} \tilde{A}_{[p]} \otimes W(F, K^{\pm 1}, H^{\pm 1})_{[l]} \cong U_q(F, K^{\pm 1}, H^{\pm 1})_{[n]}.$$  

**Proof:** First of all, the map $\tilde{A} \times W(F, K^{\pm 1}, H^{\pm 1}) \longrightarrow U_q(F, K^{\pm 1}, H^{\pm 1})$ is bilinear. So by the universal property of the tensor product, there is a map from $\tilde{A} \otimes W(F, K^{\pm 1}, H^{\pm 1})$ into $U_q(F, K^{\pm 1}, H^{\pm 1})$ defined by the multiplication. It is easy to check this map is a homomorphism of right $W(F, K^{\pm 1}, H^{\pm 1})$–modules and is surjective as well. Now we show that it is injective. Let $u \in \bigoplus_{p+i+m=nm} \tilde{A}_{[p]} \otimes W(F, K^{\pm 1}, H^{\pm 1})_{[l]}$ with $\Phi(u) = 0$. Up to an automorphism of $W(F, K^{\pm 1}, H^{\pm 1})$, we can write $u = \sum_i u_i(K, H) \otimes F^i$ where $u_i(K, H)$ are Laurent polynomials in $\mathbb{C}[K^{\pm 1}, H^{\pm 1}]$. Then $\Phi(u) = \sum u_i(K, H) F^i = 0$, so $u_i(K, H) = 0$ for all $i$. Thus $u = 0$. So we have proved that $\Phi$ is an isomorphism of vector spaces. In addition, by counting the degrees of both sides, we also have

$$\bigoplus_{p+i+m=nm} \tilde{A}_{[p]} \otimes W(F, K^{\pm 1}, H^{\pm 1})_{[l]} \cong U_q(F, K^{\pm 1}, H^{\pm 1})_{[n]}.$$  

Thus we have proved the theorem.  

Let $Y_\eta$ be the left $U_q(f(K, H))$–module defined by

$$Y_\eta = U_q(f(K, H)) \otimes_{U_q(E)} \mathbb{C}_\eta,$$

where $\mathbb{C}_\eta$ is the one dimensional $U_q(E)$–module defined by the character $\eta$. It is easy to see that

$$Y_\eta = U_q(f(K, H))/U_q(f(K, H))U_{q, \eta}(E)$$
So we are done. Let us recall Lemma 4.7.

We need an auxiliary Lemma:

**Lemma 4.8.** Let \( W = \) a Whittaker module with a cyclic vector \( 1 \). Now we have a quotient map from \( U_q(f(K, H)) \) to \( Y_\eta \) as follows:

\[
U_q(f(K, H)) \longrightarrow Y_\eta \quad \text{is defined by} \quad u \mapsto u1_\eta,
\]

for any \( u \in U_q(f(K, H)) \).

If \( u \in U_q(f(K, H)) \), then there is a \( u' \) which is the unique element in \( U_q(f(K^{\pm 1}, H^{\pm 1})) \) such that \( u1_\eta = u'1_\eta \). As in [12], we define the \( \eta \)-reduced action of \( U_q(E) \) on \( U_q(f(K^{\pm 1}, H^{\pm 1})) \) as follows:

\[
x \cdot v = (xv)^\eta - \eta(x)v,
\]

where \( x \in U_q(E) \) and \( v \in U_q(f(K^{\pm 1}, H^{\pm 1})) \).

**Lemma 4.6.** Let \( u \in U_q(f(K, H)) \) and \( x \in U_q(E) \), we have

\[
x \cdot u^\eta = [x, u]^\eta.
\]

**Proof:**

\[
[x, u]1_\eta = (xu - ux)1_\eta = (x - \eta(x)u)1_\eta.
\]

Hence

\[
[x, u]^\eta = (xu)^\eta - \eta(x)u^\eta = (xu)^\eta - \eta(x)u^\eta = x \cdot u^\eta.
\]

\[\square\]

**Lemma 4.7.** Let \( x \in U_q(E) \), \( u \in U_q(f(K^{\pm 1}, H^{\pm 1})) \), and \( v \in W(E, K^{\pm 1}, H^{\pm 1}) \), then we have

\[
x \cdot (uv) = (x \cdot u)v.
\]

**Proof:** Let \( v = w^\eta \) for some \( w \in Z(U_q(f(K, H))) \), then \( uw = uw^\eta = u^\eta w^\eta = (uw)^\eta \). Thus

\[
x \cdot (uv) = x \cdot (uw)^\eta = [x, uw]^\eta
\]

\[
= ([x, u]w)^\eta = [x, u]^\eta w^\eta = (x \cdot u^\eta)v
\]

\[
= (x \cdot u)v.
\]

So we are done. \[\square\]

Let \( V \) be an \( U_q(f(K, H)) \)-module and let \( U_{q,V}(f(K, H)) \) be the annihilator of \( V \) in \( U_q(f(K, H)) \). Then \( U_{q,V}(f(K, H)) \) defines a central ideal \( Z_V \subset Z(U_q(f(K, H))) \) by setting \( Z_V = U_{q,V}(f(K, H)) \cap Z(U_q(f(K, H))) \). Suppose that \( V \) is a Whittaker module with a cyclic Whittaker vector \( w \), we denote by \( U_{q,w}(f(K, H)) \) the annihilator of \( w \) in \( U_q(f(K, H)) \). It is obvious that

\[
U_q(f(K, H))U_{q,E}(E) + U_q(f(K, H))Z_V \subset U_{q,w}(f(K, H)).
\]

In the next theorem, we show that the reverse inclusion holds. First of all, we need an auxiliary Lemma:

**Lemma 4.8.** Let \( X = \{ v \in U_q(F, K^{\pm 1}, H^{\pm 1}) \mid (x \cdot v)w = 0, x \in U_q(E) \} \). Then

\[
X = \hat{A} \otimes W_V(F, K^{\pm 1}, H^{\pm 1}) + W(F, K^{\pm 1}, H^{\pm 1}),
\]

where \( W_V(F, K^{\pm 1}, H^{\pm 1}) = (Z_V)^\eta \). In fact, \( U_{q,V}(F, K^{\pm 1}, H^{\pm 1}) \subset X \) and

\[
U_{q,w}(F, K^{\pm 1}, H^{\pm 1}) = \hat{A} \otimes W_w(F, K^{\pm 1}, H^{\pm 1}),
\]

where \( U_{q,w}(F, K^{\pm 1}, H^{\pm 1}) = U_{q,w}(f(K, H)) \cap U_q(F, K^{\pm 1}, H^{\pm 1}) \).
Proof: Let us denote by \( Y = \check{A} \otimes W_V(F, K^{q+1}, H^{q+1}) + W(F, K^{q+1}, H^{q+1}) \) where \( W(F, K^{q+1}, H^{q+1}) = (Z(U_q(f(K, H))))^q \). Thus we need to verify \( X = Y \). Let \( v \in W(F, K^{q+1}, H^{q+1}) \), then \( v = u^q \) for some \( u \in Z(U_q(f(K, H))) \). So we have
\[
x \bullet v = x \bullet u^q = [x, u]^q = (xu)^q - \eta(x)u^q = x^q u^q - \eta(x)u^q = 0.
\]
So we have \( W(F, K^{q+1}, H^{q+1}) \subset X \). Let \( u \in Z_V \) and \( v \in U_q(F, K^{q+1}, H^{q+1}) \). Then for any \( x \in U_q(E) \), we have
\[
x \bullet (vu)^q = (x \bullet v)u^q.
\]
Since \( u \in Z_V \), then \( u^q \in U_q(f(K, H)) \). Thus we have \( vu^q \in X \), hence
\[
\check{A} \otimes W_V(F, K^{q+1}, H^{q+1}) \subset X,
\]
which proves \( Y \subset X \). Let \( A_{i[q]} \) be the subspace of \( \mathbb{C}[K^{q+1}, H^{q+1}] \) spanned by \( K^{q+1}, H^{q+1} \), and let \( W_V(F, K^{q+1}, H^{q+1}) \) be the complement of \( W_V(F, K^{q+1}, H^{q+1}) \) in \( W(F, K^{q+1}, H^{q+1}) \).

Let us set
\[
M_i = \check{A}_{i[q]} \otimes W_V(F, K^{q+1}, H^{q+1}),
\]
thus we have the following:
\[
U_q(F, K^{q+1}, H^{q+1}) = M \oplus Y,
\]
where \( M = \sum_{i \geq 1} M_i \). We show that \( M \cap X \neq 0 \). Let \( M_{[k]} = \sum_{1 \leq i \leq k} M_i \), then \( M_{[k]} \) is a filtration of \( M \). Suppose \( n \) is the smallest integer such that \( X \cap M_{[n]} \neq 0 \) and \( 0 \neq y \in X \cap M_{[n]} \). Then we have \( y = \sum_{1 \leq i \leq n} y_i \) where \( y_i \in \check{A}_{i[q]} \otimes W_V(F, K^{q+1}, H^{q+1}) \).

Suppose we have chosen \( y \) in such a way that \( y \) has the fewest terms. By similar computations as in [15], we have \( 0 \neq y - \frac{1}{n} \sum_{1 \leq i \leq n} E \bullet y \in X \cap M_{[n]} \) with fewer terms than \( y \). This is a contradiction. So we have \( X \cap M = 0 \).

Now we prove that \( U_{q, w}(F, K^{q+1}, H^{q+1}) \subset X \). Let \( u \in U_{q, w}(F, K^{q+1}, H^{q+1}) \) and \( x \in U_q(E) \), then we have \( xuw = 0 \) and \( uxw = \eta(x)uw = 0 \). Thus \( [x, u] \in U_{q, w}(f(K, H)) \), hence \( [x, u]^q \in U_{q, w}(F, K^{q+1}, H^{q+1}) \). Since \( u \in U_{q, w}(F, K^{q+1}, H^{q+1}) \subset U_{q, w}(E, F, K^{q+1}, H^{q+1}) \), then \( x \bullet u = [x, u]^q \). Thus \( x \bullet u \in U_{q, w}(F, K^{q+1}, H^{q+1}) \). So \( u \in X \) by the definition of \( X \). Now we are going to prove the following:
\[
W(F, K^{q+1}, H^{q+1}) \cap U_{q, w}(F, K^{q+1}, H^{q+1}) = W_V(F, K^{q+1}, H^{q+1}).
\]
In fact, \( W_V(F, K^{q+1}, H^{q+1}) = (Z_V^q) \) and \( W_V(F, K^{q+1}, H^{q+1}) \subset U_{q, w}(F, K^{q+1}, H^{q+1}) \). So if \( v \in W(F, K^{q+1}, H^{q+1}) \cap U_{q, w}(F, K^{q+1}, H^{q+1}) \), then we can uniquely write \( v = u^q \) for \( u \in Z(U_q(f(K, H))) \). Then \( vu = 0 \) implies \( uw = 0 \) and hence \( u \in Z(U_q(f(f(K, H)))) \). Since \( V \) is generated cyclically by \( w \), we have proved the above statement. Obviously, we have \( U_q(f(K, H))Z_V \subset U_{q, w}(f(K, H)) \).

Thus we have \( \check{A} \otimes W_V(F, K^{q+1}, H^{q+1}) \subset U_{q, w}(F, K^{q+1}, H^{q+1}) \). Therefore, we have
\[
U_{q, w}(F, K^{q+1}, H^{q+1}) = \check{A} \otimes W_V(F, K^{q+1}, H^{q+1}).
\]
So we have finished the proof. \( \square \)
Theorem 4.2. Let $V$ be a Whittaker module admitting a cyclic Whittaker vector $w$, then we have

$$ U_{q,w}(f(K,H)) = U_q(f(K,H))Z_V + U_q(f(K,H))U_{q,\eta}(E). $$

**Proof:** It is obvious that

$$ U_q(f(K,H))Z_V + U_q(f(K,H))U_{q,\eta}(E) \subset U_{q,w}(f(K,H)). $$

Let $u \in U_{q,w}(f(K,H))$, we show that $u \in U_q(f(K,H))Z_V + U_q(f(K,H))U_{q,\eta}(E)$. Let $v = u^n$, then it suffices to show that $v \in \tilde{A} \otimes W_V(F,K^{\pm 1},H^{\pm 1})$. But $v \in U_{q,w}(F,K^{\pm 1},H^{\pm 1}) = \tilde{A} \otimes W_V(F,K^{\pm 1},H^{\pm 1})$. So we have proved the theorem. \(\Box\)

Theorem 4.3. Let $V$ be any Whittaker module for $U_q(f(K,H))$, then the correspondence

$$ V \mapsto Z_V $$

sets up a bijection between the set of all equivalence classes of Whittaker modules and the set of all ideals of $Z(U_q(f(K,H)))$.

**Proof:** Let $V_i, i = 1, 2$ be two Whittaker modules. If $Z_{V_i} = Z_{V_2}$, then clearly $V_1$ is equivalent to $V_2$ by the above Theorem. Now let $Z_i$ be an ideal of $Z(U_q(f(K,H)))$ and let $L = U_q(f(K,H))Z_i + U_q(f(K,H))U_{q,\eta}(E)$. Then $V = U_q(f(K,H))/L$ is a Whittaker module with a cyclic Whittaker vector $w = 1$. Obviously we have $U_{q,w}(f(K,H)) = L$. So $L = U_{q,w}(f(K,H)) = U_q(f(K,H))Z_V + U_q(f(K,H))U_{q,\eta}(E)$. This implies that

$$ \pi(Z_*) = \pi(L) = \pi(Z_V). $$

Since $\pi$ is injective on $Z(U_q(f(K,H)))$, thus $Z_V = Z_*$. Thus we finished the proof. \(\Box\)

Theorem 4.4. Let $V$ be an $U_q(f(K,H))$–module. Then $V$ is a Whittaker module if and only if

$$ V \cong U_q(f(K,H)) \otimes_{Z(U_q(f(K,H))))} U_q(F) (Z(U_q(f(K,H))))/Z_\eta. $$

In particular, in such a case the ideal $Z_*$ is uniquely determined to be $Z_V$.

**Proof:** Let $1_*$ be the image of 1 in $Z(U_q(f(K,H))))/Z_\eta$, then

$$ \text{Ann}_{Z(U_q(f(K,H))))} U_q(F)(1_*) = U_q(E)Z_* + Z(U_q(f(K,H))))U_{q,\eta}(E) $$

Thus the annihilator of $w = 1 \otimes 1_*$ is

$$ U_{q,w}(f(K,H)) = U_q(f(K,H))Z_* + U_q(f(K,H))U_{q,\eta}(E) $$

Then the result follows from the last theorem. \(\Box\)

Theorem 4.5. Let $V$ be an $U_q(f(K,H))$–module with a cyclic Whittaker vector $w \in V$. Then any $v \in V$ is a Whittaker vector if and only if $v = uw$ for some $u \in Z(U_q(f(K,H))))$.

**Proof:** If $v = uw$ for some $u \in Z(U_q(f(K,H))))$, then it is obvious that $v$ is a Whittaker vector. Conversely, let $v = uw$ for some $u \in U_q(f(K,H))$ be a Whittaker vector of $V$. Then $v = uw$ by the definition of Whittaker module. So we may assume that $u \in U_q(F,K^{\pm 1},H^{\pm 1})$. If $x \in U_q(E)$, then we have $xuw = \eta(x)uw$ and $uxw = \eta(x)uw$. Thus $[x,u]w = 0$ and hence $[x,u]w = 0$. But we have $x \cdot u = [x,u]w$. Thus we have $u \in X$. We can now write $u = u_1 + u_2$ with $u_1 \in U_{q,w}(F,K^{\pm 1},H^{\pm 1})$ and $u_2 \in W(F,K^{\pm 1},H^{\pm 1})$. Then $u_1w = 0$. Hence $u_2w = v$. \(\Box\)
But \( u_2 = u_3^3 \) with \( u_3 \in Z(U_q(f(K, H))) \). So we have \( v = u_3w \) which proves the theorem.

Now let \( V \) be a Whittaker module and \( \text{End}_{U_q(f(K, H))}(V) \) be the endomorphism ring of \( V \) as a \( U_q(f(K, H)) \)-module. Then we can define the following homomorphism of algebras using the action of \( Z(U_q(f(K, H))) \) on \( V \):

\[
\pi_V : Z(U_q(f(K, H))) \rightarrow \text{End}_{U_q(f(K, H))}(V).
\]

It is clear that

\[
Z(U_q(f(K, H)))/Z(V(U_q(f(K, H)))) \cong \pi_V(Z(U_q(f(K, H)))) \subset \text{End}_{U_q(f(K, H))}(V).
\]

In fact, the next theorem says that this inclusion is an equality as well.

**Theorem 4.6.** Let \( V \) be a Whittaker \( U_q(f(K, H)) \)-module. Then \( \text{End}_{U_q(f(K, H))}(V) \cong Z(U_q(f(K, H)))/Z(V) \). In particular, \( \text{End}_{U_q(f(K, H))}(V) \) is commutative.

**Proof:** Let \( w \in V \) be a cyclic Whittaker vector. If \( \alpha \in \text{End}_{U_q(f(K, H))}(V) \), then \( \alpha(w) = uw \) for some \( u \in Z(U_q(f(K, H))) \) by Theorem 4.5. Thus we have \( \alpha(ww) = uvw = uvw, \) hence \( \alpha = \pi_u \), which proves the theorem.

Now we are going to construct explicitly some Whittaker modules. Let

\[
\xi : Z(U_q(f(K, H))) \rightarrow \mathbb{C}
\]

be a central character of the center \( Z(U_q(f(K, H))) \). For any given central character \( \xi \), let \( Z_\xi = \text{Ker}(\xi) \subset Z(U_q(f(K, H))) \) and \( Z_\xi \) is a maximal ideal of \( Z(U_q(f(K, H))) \). Since \( \mathbb{C} \) is algebraically closed, then \( Z_\xi = (\Omega - a_\xi, KH - b_\xi) \) for some \( a_\xi \in \mathbb{C}, b_\xi \in \mathbb{C}^* \).

For any given central character \( \xi \), let \( \mathcal{C}_{\xi, \eta} \) be the one dimensional \( Z(U_q(f(K, H))) \otimes U_q(E) \)-module defined by \( uvy = \xi(u)\eta(v)y \) for any \( u \in Z(U_q(f(K, H))) \) and any \( v \in U_q(E) \). We set

\[
Y_{\xi, \eta} = U_q(f(K, H)) \otimes_{Z(U_q(f(K, H))) \otimes U_q(E)} \mathcal{C}_{\xi, \eta}.
\]

It is easy to see that \( Y_{\xi, \eta} \) is a Whittaker module of type \( \eta \) and admits a central character \( \xi \). By Schur’s lemma, we know every irreducible representation has a central character. As studied in \([10]\), we know \( U_q(f(K, H)) \) has a similar theory for Verma modules. In fact, Verma modules also fall into the category of Whittaker modules if we take the trivial character of \( U_q(E) \). Namely we have the following

\[
M_\lambda = U_q(f(K, H)) \otimes_{U_q(E, K^{\pm 1}, H^{\pm 1})} \mathcal{C}_\lambda,
\]

where \( K, H \) act on \( \mathcal{C}_\lambda \) through the character \( \lambda \) of \( \mathbb{C}[K^{\pm 1}, H^{\pm 1}] \) and \( U_q(E) \) act trivially on \( \mathcal{C}_\lambda \). Thus, \( M_\lambda \) admits a central character. It is well-known that Verma modules may not be necessarily irreducible, even though they have central characters. However, Whittaker modules are in the other extreme as shown in the next theorem:

**Theorem 4.7.** Let \( V \) be a Whittaker module for \( U_q(f(K, H)) \). Then the following statements are equivalent.

1. \( V \) is irreducible.
2. \( V \) admits a central character.
3. \( Z_V \) is a maximal ideal.
(4) The space of Whittaker vectors of $V$ is one-dimensional.

(5) All nonzero Whittaker vectors of $V$ are cyclic.

(6) The centralizer $\text{End}_{U_q(F,K,H)}(V)$ is reduced to $\mathbb{C}$.

(7) $V$ is isomorphic to $Y_{\xi,\eta}$ for some central character $\xi$.

Proof: It is easy to see that (2) – (7) are equivalent to each other by using the previous Theorems we have just proved. Since $\mathbb{C}$ is algebraically closed and uncountable, we also know (1) implies (2) by using a theorem due to Dixmier ([3]). To complete the proof, it suffices to show that (2) implies (1), namely if $V$ has a central character, then $V$ is irreducible. Let $\omega \in V$ be a cyclic Whittaker vector, then $V = U_q(f(K,H))\omega$. Since $V$ has a central character, then it is easy to see from the description of the center that $V$ has a $\mathbb{C}$-basis consisting of elements $\{K^i\omega, H^j\omega \mid i, j \in \mathbb{Z}_{\geq 0}\}$. Let $0 \neq v = (\sum_{i=0}^{n} a_i K^i + \sum_{j=1}^{m} b_j H^j)\omega \in V$, then $E(\sum_{i=0}^{n} a_i K^i + \sum_{j=1}^{m} b_j H^j)\omega = (\sum_{i=0}^{n} q^{-2i} a_i K^i + \sum_{j=1}^{m} q^{2j} b_j H^j)\omega$. Thus we have $0 \neq \xi q^{-2n}v - Ev \in V$, in which the top degree of $K$ is $n-1$. By repeating this operation finitely many times, we will finally get an element $0 \neq aw$ with $a \in \mathbb{C}^*$. This means that $V = U_q(f(K,H))v$ for any $0 \neq v \in V$. So $V$ is irreducible. Therefore, we are done with the proof.

It is easy to show the following two theorems, for more details about the proof, we refer the reader to [12].

Theorem 4.8. Let $V$ be a $U_q(f(K,H))$-module which admits a central character. Assume that $w \in V$ is a Whittaker vector. Then the submodule $U_q(f(K,H))w \subseteq V$ is irreducible.

Theorem 4.9. Let $V_1,V_2$ be any two irreducible $U_q(f(K,H))$-modules with the same central character. If $V_1$ and $V_2$ contain Whittaker vectors, then these vectors are unique up to scalars. And furthermore, $V_1$ and $V_2$ are isomorphic to each other as $U_q(f(K,H))$-modules.

In fact we have the following description about the basis of an irreducible Whittaker module $(V,w)$ where $w \in V$ is a cyclic Whittaker vector.

Theorem 4.10. Let $(V,w)$ be an irreducible Whittaker module with a Whittaker vector $w$, then $V$ has a $\mathbb{C}$-basis consisting of elements $\{K^i\omega, H^j\omega \mid i, j \in \mathbb{Z}_{\geq 0}\}$.

Proof: Since $w$ is a cyclic Whittaker vector of the Whittaker module $V$, then we have $V = U_q(f(K,H))w$. Since $U_q(f(K,H))w = U_q(F, K^\pm, H^\pm)\otimes U_q(E)\mathbb{C}_w$, then we have $V = U_q(F, K^\pm, H^\pm)w$. Since $V$ is irreducible, then $V$ has a central character. Thus we have $\Omega w = \lambda(\Omega)w$. Now $\Omega w = (\eta(E)F + \frac{q^{2m}K^m + H^m}{(q^m-1)(q^{-m}-1)})w$. Hence the action of $F$ on $V$ is uniquely determined by the action of $K$ and $H$ on $V$, and $H^{-1}v = aKv, K^{-1}v = bHv$ for some $a, b \in \mathbb{C}^*$ and for any $v \in V$. Thus the theorem follows.

References