Chapter 5 Polar Coordinates; Vectors
5.1 Polar coordinates
1. Pole and polar axis

2. Polar coordinates
A point $P$ in a polar coordinate system is represented by an ordered pair of numbers $(r, \theta)$. If $r > 0$, then $r$ is the distance of the point from the pole; $\theta$ (in degrees or radians) formed by the polar axis and a ray from the pole through the point. We call the ordered pair $(r, \theta)$ the polar coordinate of the point.

3. Example: In polar coordinate system, locate point $(2, \frac{\pi}{4})$.

4. In using polar coordinates $(r, \theta)$, if $r < 0$, the corresponding point is on the ray from the pole extending in the direction opposite the terminal side of $\theta$ at a distance $|r|$ units from the pole.
5. Example: In polar coordinate system, locate point \((-3, \frac{2\pi}{3})\).

6. Example: Plot the points with the following polar coordinates:
   \(1\) \((3, \frac{5\pi}{3})\) \hspace{1cm} \(2\) \((-2, -\frac{\pi}{4})\) \hspace{1cm} \(3\) \((3, 0)\) \hspace{1cm} \(4\) \((-2, \frac{\pi}{4})\)

7. Example: Find several polar coordinates of a single point \((2, \frac{\pi}{4})\).

8. Remark: A point with polar coordinates \((r, \theta)\) also can be represented by either of the following:

   \[ (r, \theta + 2k\pi) \quad \text{or} \quad (-r, \theta + \pi + 2k\pi), \quad k \text{ any integer} \]

   The polar coordinates of the pole are \((0, \theta)\), where \(\theta\) can be any angle.

9. Example: Plot the point with polar coordinates \((3, \frac{\pi}{6})\), and find other polar coordinates \((r, \theta)\) of this same point for which:
   \(1\) \(r > 0, 2\pi \leq \theta \leq 4\pi\)
   
   \(2\) \(r < 0, 0 \leq \theta < 2\pi\)
(3) $r > 0, -2\pi \leq \theta < 0$

10. Theorem: Conversion from polar coordinates to rectangular coordinates

If $P$ is a point with polar coordinates $(r, \theta)$, the rectangular coordinates $(x, y)$ of $P$ are given by

$$x = r \cos \theta \quad y = r \sin \theta$$

11. Example: Find the rectangular coordinates of the points with the following polar coordinates:

(1) $(6, \frac{\pi}{6})$  
(2) $(-4, -\frac{\pi}{4})$.

12. Converting from rectangular coordinates to polar coordinates:

Case 1: Points that lie on either the $x$-axis or the $y$-axis. Suppose $a > 0$, we have
Case 2: Points that lie in a quadrant.

13. Summary: Steps for converting from rectangular to polar coordinates.
   (1) Always plot the point \((x, y)\) first.
   (2) To find \(r\) using \(r^2 = x^2 + y^2\).
   (3) To find \(\theta\), first determine the quadrant that the point lies in:

   Quadrant I: \(\theta = \tan^{-1} \frac{y}{x}\)  
   Quadrant II: \(\theta = \pi + \tan^{-1} \frac{y}{x}\)  
   Quadrant III: \(\theta = \pi + \tan^{-1} \frac{y}{x}\)  
   Quadrant IV: \(\theta = \tan^{-1} \frac{y}{x}\)

14. Find polar coordinates of a point whose rectangular coordinates are:
   (1) \((0, 3)\)
   (2) \((2, -2)\)
   (3) \((-1, -\sqrt{3})\)
15. Example: Transform the equation \( r = 4 \sin \theta \) from polar coordinates to rectangular coordinates.

16. Example: Transform the equation \( 4xy = 9 \) from rectangular coordinates to polar coordinates.
5.2 Polar equations and graphs
1. Polar grids consist of concentric circles (with centers at the pole) and rays (with vertices at the pole).

2. The graph of a polar equation
   An equation whose variables are polar coordinates is called a polar equation.
   The graph of a polar equation consists of all points whose polar coordinates satisfy the equation.

3. Example: Identify and graph the equation: \( r = 3 \).
4. Example: Identify and graph the equation:
$$\theta = \frac{\pi}{4}.$$ 

5. Identify and graph the equation: $r \sin \theta = 2.$
6. Identify and graph the equation: \( r \cos \theta = -3 \).

7. Theorem: Let \( a \) be a nonzero real number. Then the graph of the equation

\[
r \sin \theta = a
\]

is a horizontal line \( a \) units above the pole if \( a > 0 \) and \( |a| \) units below the pole if \( a < 0 \).

The graph of the equation

\[
r \cos \theta = a
\]

is a vertical line \( a \) units to the right of the pole if \( a > 0 \) and \( |a| \) units to the left of the pole if \( a < 0 \).
8. Example: Identify and graph the equation: \( r = 4 \sin \theta \).

9. Example: Identify and graph the equation: \( r = -2 \cos \theta \).
10. Theorem: Let $a$ be a positive real number. Then the graph of each of the following equation is a circle passing through the pole:

(1) $r = 2a \sin \theta$: radius $a$; center at $(0, a)$ in rectangular coordinates

(2) $r = -2a \sin \theta$: radius $a$; center at $(0, -a)$ in rectangular coordinates

(3) $r = 2a \cos \theta$: radius $a$; center at $(a, 0)$ in rectangular coordinates

(4) $r = -2a \cos \theta$: radius $a$; center at $(-a, 0)$ in rectangular coordinates
5.3 The Complex Plane; De Moivre’s Theorem

1. Complex plane: Real axis and imaginary axis

2. Remark:
   Any point on the real axis is of the form \( z = x + 0i = x \), a real number.
   Any point on the imaginary axis is of the form \( z = 0 + yi \), a pure imaginary number.

3. Definition: Let \( z = x + yi \) be a complex number. The \underline{magnitude or modulus} of \( z \) denoted by \(|z|\), is defined as the distance from the origin to the point \((x, y)\). That is,
   \[
   |z| = \sqrt{x^2 + y^2}
   \]

4. Recall that if \( z = x + yi \), then its \underline{conjugate}, denoted by \( \bar{z} = x - yi \).
   Example:

   It follows from \( z\bar{z} = x^2 + y^2 \) that \( |z| = \sqrt{z\bar{z}} \).
5. Polar form of a complex number: When a complex number is written in the standard form $z = x + yi$, we say that it is in rectangular form, or Cartesian form.

If $r \geq 0$ and $0 \leq \theta < 2\pi$, the complex number $z = x + yi$ may be written in polar form as

$$z = x + yi = (r \cos \theta) + (r \sin \theta)i = r(\cos \theta + i \sin \theta)$$

If $z = r(\cos \theta + i \sin \theta)$ is the polar form of a complex number, the angle $\theta$, $0 \leq \theta < 2\pi$, is called the argument of $z$.

Because $r \geq 0$, we have $r = \sqrt{x^2 + y^2}$, the magnitude of $z = r(\cos \theta + i \sin \theta)$ is $|z| = r$.

6. Example: Plot the point corresponding to $z = \sqrt{3} - i$ in the complex plane, and write an expression for $z$ in polar form.

7. Example: Plot the point corresponding to $z = 2(\cos 30^0 + i \sin 30^0)$ in the complex plane, and write an expression for $z$ in rectangular form.
8. Theorem: Let \( z_1 = r_1(\cos \theta_1 + i \sin \theta_1) \) and \( z_2 = r_2(\cos \theta_2 + i \sin \theta_2) \) be two complex numbers. Then
\[
z_1 z_2 = r_1 r_2 [\cos(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2)]
\]
If \( z_2 \neq 0 \), then
\[
\frac{z_1}{z_2} = \frac{r_1}{r_2} [\cos(\theta_1 - \theta_2) + i \sin(\theta_1 - \theta_2)]
\]

9. Example: If \( z = 3(\cos 20^0 + i \sin 20^0) \) and \( w = 5(\cos 100^0 + i \sin 100^0) \), find \( zw \) and \( \frac{z}{w} \).

10. Theorem: De Moivre’s Theorem

If \( z = r(\cos \theta + i \sin \theta) \) is a complex number, then
\[
z^n = r^n [\cos(n\theta) + i \sin(n\theta)]
\]
where \( n \geq 1 \) is a positive integer.

11. Example: Write \( [2(\cos 20^0 + i \sin 20^0)]^3 \) in the standard form \( a + bi \).
12. Example: Write $(1 + i)^5$ in the standard form $a + bi$.

13. Definition: Let $w$ be a given complex number, and let $n \geq 2$ denote a positive integer. Any complex number $z$ that satisfies the equation $z^n = w$ is called a complex $n$th root of $w$.

14. Theorem: Let $w = r(\cos \theta_0 + i \sin \theta_0)$ be a complex number and let $n \geq 2$ be an integer. If $w \neq 0$, there are $n$ distinct complex roots of $w$, given by the formula

$$z_k = \sqrt[n]{r}[\cos(\frac{\theta_0}{n} + \frac{2k\pi}{n}) + i \sin(\frac{\theta_0}{n} + \frac{2k\pi}{n})]$$

where $k = 0, 1, 2, \ldots, n - 1$.

15. Example: Find the complex cube roots of $-1 + \sqrt{3}i$. 


5.4 Vectors

1. Definition: A vector is a quantity that has both magnitude and direction.

   It is customary to present a vector by using an arrow. The length of the arrow represents the magnitude of the vector, and the arrowhead indicates the direction of the vector.

2. Definition: If $P$ and $Q$ are two distinct points in the $xy$-plane, there is exactly one line containing both $P$ and $Q$. The points on that part of the line that joins $P$ to $Q$, including $P$ and $Q$, form what is called the line segment $PQ$. If we order the points so that they proceed from $P$ to $Q$, we have a directed line segment from $P$ to $Q$, or a geometric vector, which we denote by $PQ$. We call $P$ the initial point and $Q$ the terminal point.
3. The magnitude of the directed line segment $\overrightarrow{PQ}$ is the distance from the point $P$ to the point $Q$; that is the length of the line segment. The direction of $\overrightarrow{PQ}$ is from $P$ to $Q$. If a vector $\overrightarrow{v}$ has the same magnitude and the same direction as the directed line segment $\overrightarrow{PQ}$, we write $\overrightarrow{v} = \overrightarrow{PQ}$.

The vector $\overrightarrow{v}$ whose magnitude is 0 is called the zero vector, $\overrightarrow{0}$. The zero vector is assigned no direction.

Two vectors $\overrightarrow{v}$ and $\overrightarrow{w}$ are equal, written $\overrightarrow{v} = \overrightarrow{w}$ if they have the same magnitude and the same direction.

4. Definition: The sum $\overrightarrow{v} + \overrightarrow{w}$ of two vectors is defined as follows: We position the vectors $\overrightarrow{v}$ and $\overrightarrow{w}$ so that the terminal point of $\overrightarrow{v}$ coincides with the initial point of $\overrightarrow{w}$. The vector $\overrightarrow{v} + \overrightarrow{w}$ is then the unique vector whose initial point coincides with the initial point of $\overrightarrow{v}$ and whose terminal point coincides with the terminal point of $\overrightarrow{w}$.

5. Properties of vector addition:

   (1) Vector addition is commutative: $\overrightarrow{v} + \overrightarrow{w} = \overrightarrow{w} + \overrightarrow{v}$
(2) Vector addition is associative: \( \overrightarrow{u} + (\overrightarrow{v} + \overrightarrow{w}) = (\overrightarrow{u} + \overrightarrow{v}) + \overrightarrow{w} \)

(3) \( \overrightarrow{v} + \overrightarrow{0} = \overrightarrow{0} + \overrightarrow{v} = \overrightarrow{v} \)

(4) If \( \overrightarrow{v} \) is a vector, then \( -\overrightarrow{v} \) is the vector having the same magnitude as \( \overrightarrow{v} \) but whose direction is opposite to \( \overrightarrow{v} \).

(5) \( \overrightarrow{v} + (-\overrightarrow{v}) = (-\overrightarrow{v}) + \overrightarrow{v} = \overrightarrow{0} \)

6. Definition: If \( \overrightarrow{v} \) and \( \overrightarrow{w} \) are two vectors, we define the difference \( \overrightarrow{v} - \overrightarrow{w} \) as \( \overrightarrow{v} - \overrightarrow{w} = \overrightarrow{v} + (-\overrightarrow{w}) \)
7. Multiplying vectors by numbers

If \( \alpha \) is a scalar and \( \vec{v} \) is a vector, the scalar product \( \alpha \vec{v} \) is defined as follows:

1. If \( \alpha > 0 \), the product \( \alpha \vec{v} \) is the vector whose magnitude is \( \alpha \) times the magnitude of \( \vec{v} \) and whose direction is the same as \( \vec{v} \).

2. If \( \alpha < 0 \), the product \( \alpha \vec{v} \) is the vector whose magnitude is \( |\alpha| \) times the magnitude of \( \vec{v} \) and whose direction is opposite that of \( \vec{v} \).

3. If \( \alpha = 0 \) or if \( \vec{v} = \vec{0} \), then \( \alpha \vec{v} = \vec{0} \)

Properties of scalar multiplication:

\[
0 \vec{v} = \vec{0}, \quad 1 \vec{v} = \vec{v}, \quad -1 \vec{v} = -\vec{v}
\]

\[
(\alpha + \beta) \vec{v} = \alpha \vec{v} + \beta \vec{v}, \quad \alpha(\vec{v} + \vec{w}) = \alpha \vec{v} + \alpha \vec{w}
\]

\[
\alpha(\beta \vec{v}) = (\alpha\beta) \vec{v}
\]

8. Example:
9. Magnitudes of vectors: If $\overrightarrow{v}$ is a vector, we use the symbol $\|\overrightarrow{v}\|$ to represent the magnitude of $\overrightarrow{v}$.

10. Theorem: Properties of $\|\overrightarrow{v}\|$ 

If $\overrightarrow{v}$ is a vector and if $\alpha$ is a scalar, then 

(1) $\|\overrightarrow{v}\| \geq 0$,  
(2) $\|\overrightarrow{v}\| = 0$ if and only if $\overrightarrow{v} = \overrightarrow{0}$  
(3) $\| - \overrightarrow{v}\| = \|\overrightarrow{v}\|$  
(4) $\|\alpha \overrightarrow{v}\| = |\alpha|\|\overrightarrow{v}\|$  
(5) A vector $\overrightarrow{u}$ for which $\|\overrightarrow{u}\| = 1$ is called a unit vector

11. Definition: An algebraic vector is represented as 

$$\overrightarrow{v} = \langle a, b \rangle$$

where $a$ and $b$ are real numbers (scalars) called the components of the vector $\overrightarrow{v}$.

We use a rectangular coordinate system to represent algebraic vectors in the plane. If $\overrightarrow{v} = \langle a, b \rangle$ is an algebraic vector whose initial point is at the origin, then $\overrightarrow{v}$ is called a position vector. Note that the terminal point of the position vector $\overrightarrow{v} = \langle a, b \rangle$ is $P = (a, b)$.

12. Theorem: Suppose that $\overrightarrow{v}$ is a vector with initial point $P_1 = (x_1, y_1)$, not necessarily the origin, and terminal point $P_2 = (x_2, y_2)$. If $\overrightarrow{v} = P_1P_2$, then $\overrightarrow{v}$ is equal to the position vector

$$\overrightarrow{v} = (x_2 - x_1, y_2 - y_1)$$
13. Example: Find the position vector of the vector \( \overrightarrow{v} = \overrightarrow{P_1P_2} \) if \( P_1 = (-1, 2) \) and \( P_2 = (4, 6) \).

14. Theorem: Equality of vectors

Two vectors \( \overrightarrow{v} \) and \( \overrightarrow{w} \) are equal if and only if their corresponding components are equal. That is,

\[
\text{If } \overrightarrow{v} = \langle a_1, b_1 \rangle \text{ and } \overrightarrow{w} = \langle a_2, b_2 \rangle \\
\text{then } \overrightarrow{v} = \overrightarrow{w} \text{ if and only if } a_1 = a_2 \text{ and } b_1 = b_2
\]

15. Alternative representation of a vector: Let \( \overrightarrow{i} \) denote the unit vector whose direction is along the positive \( x \)-axis; let \( \overrightarrow{j} \) denote the unit vector whose direction is along the positive \( y \)-axis. Then \( \overrightarrow{i} = \langle 1, 0 \rangle \) and \( \overrightarrow{j} = \langle 0, 1 \rangle \). Any vector \( \overrightarrow{v} = \langle a, b \rangle \) can be written using the unit vectors \( \overrightarrow{i} \) and \( \overrightarrow{j} \) as follows:

\[
\overrightarrow{v} = \langle a, b \rangle = a \langle 1, 0 \rangle + b \langle 0, 1 \rangle = a \overrightarrow{i} + b \overrightarrow{j}
\]

We call \( a \) and \( b \) the horizontal and vertical components of \( \overrightarrow{v} \), respectively.

16. Properties: Let \( \overrightarrow{v} = a_1 \overrightarrow{i} + b_1 \overrightarrow{j} = \langle a_1, b_1 \rangle \) and \( \overrightarrow{w} = a_2 \overrightarrow{i} + b_2 \overrightarrow{j} = \langle a_2, b_2 \rangle \) be two vectors, and let \( \alpha \) be a scalar. Then

\[
\begin{align*}
(1) \quad \overrightarrow{v} + \overrightarrow{w} &= (a_1 + a_2) \overrightarrow{i} + (b_1 + b_2) \overrightarrow{j} = \langle a_1 + a_2, b_1 + b_2 \rangle \\
(2) \quad \overrightarrow{v} - \overrightarrow{w} &= (a_1 - a_2) \overrightarrow{i} + (b_1 - b_2) \overrightarrow{j} = \langle a_1 - a_2, b_1 - b_2 \rangle \\
(3) \quad \alpha \overrightarrow{v} &= (\alpha a_1) \overrightarrow{i} + (\alpha b_1) \overrightarrow{j} = \langle \alpha a_1, \alpha b_1 \rangle \\
(4) \quad \|\overrightarrow{v}\| &= \sqrt{a_1^2 + b_1^2}
\end{align*}
\]
17. Example: If $\vec{v} = 2\vec{i} + 3\vec{j} = \left< 2, 3 \right>$ and $\vec{w} = 3\vec{i} - 4\vec{j} = \left< 3, -4 \right>$, find $\vec{v} + \vec{w}$ and $\vec{v} - \vec{w}$.

18. Example: If $\vec{v} = 2\vec{i} + 3\vec{j} = \left< 2, 3 \right>$ and $\vec{w} = 3\vec{i} - 4\vec{j} = \left< 3, -4 \right>$, find

   (1) $3\vec{v}$  (2) $2\vec{v} - 3\vec{w}$  (3) $\|\vec{v}\|$

19. Theorem: **Unit vector in the direction of $\vec{v}$**

   For any nonzero vector $\vec{v}$, the vector

   $$\vec{u} = \frac{\vec{v}}{\|\vec{v}\|}$$

   is a unit vector that has the same direction as $\vec{v}$.

20. Example: Find the unit vector in the same direction as $\vec{v} = 4\vec{i} - 3\vec{j}$.
5.5 The dot product

1. Definition: If $\vec{v} = a_1 \vec{i} + b_1 \vec{j}$ and $\vec{w} = a_2 \vec{i} + b_2 \vec{j}$ are two vectors, the dot product $\vec{v} \cdot \vec{w}$ is defined as

$$\vec{v} \cdot \vec{w} = a_1 a_2 + b_1 b_2.$$ 

2. Example: If $\vec{v} = 2 \vec{i} - 3 \vec{j}$ and $\vec{w} = 5 \vec{i} + 3 \vec{j}$, find:

   (1) $\vec{v} \cdot \vec{w}$    (2) $\vec{w} \cdot \vec{v}$    (3) $\vec{v} \cdot \vec{v}$    (4) $\vec{v} \cdot \vec{w}$    (5) $|\vec{v}|$    (6) $|\vec{w}|$

3. Properties of the dot product

   If $\vec{u}$, $\vec{v}$, and $\vec{w}$ are vectors, then

   (1) $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$
   (2) $\vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}$
   (3) $\vec{v} \cdot \vec{v} = |\vec{v}|^2$
   (4) $\vec{0} \cdot \vec{v} = 0$
4. Angle between vectors Let $\overrightarrow{u}$ and $\overrightarrow{v}$ are two nonzero vectors, the angle $\theta$, $0 \leq \theta \leq \pi$, between $\overrightarrow{u}$ and $\overrightarrow{v}$ is determined by the formula

$$\cos \theta = \frac{\overrightarrow{u} \cdot \overrightarrow{v}}{\|\overrightarrow{u}\| \|\overrightarrow{v}\|}$$

5. Example: Find the angle $\theta$ between $\overrightarrow{u} = 4\overrightarrow{i} - 3\overrightarrow{j}$ and $\overrightarrow{v} = 2\overrightarrow{i} + 5\overrightarrow{j}$. 
5.5 The dot product

1. Definition: If $\vec{v} = a_1 \hat{i} + b_1 \hat{j}$ and $\vec{w} = a_2 \hat{i} + b_2 \hat{j}$ are two vectors, the dot product $\vec{v} \cdot \vec{w}$ is defined as

$$\vec{v} \cdot \vec{w} = a_1 a_2 + b_1 b_2.$$ 

2. Example: If $\vec{v} = 2 \hat{i} - 3 \hat{j}$ and $\vec{w} = 5 \hat{i} + 3 \hat{j}$, find:

   (1) $\vec{v} \cdot \vec{w}$  
   (2) $\vec{w} \cdot \vec{v}$  
   (3) $\vec{v} \cdot \vec{v}$  
   (4) $\vec{v} \cdot \vec{w}$  
   (5) $\|\vec{v}\|$  
   (6) $\|\vec{w}\|$  

3. Properties of the dot product

   If $\vec{u}$, $\vec{v}$, and $\vec{w}$ are vectors, then

   (1) $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$
   (2) $\vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}$
   (3) $\vec{v} \cdot \vec{v} = \|\vec{v}\|^2$
   (4) $\vec{0} \cdot \vec{v} = 0$
4. Angle between vectors Let $\vec{u}$ and $\vec{v}$ are two nonzero vectors, the angle $\theta$, $0 \leq \theta \leq \pi$, between $\vec{u}$ and $\vec{v}$ is determined by the formula

$$\cos \theta = \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\| \|\vec{v}\|}$$

5. Example: Find the angle $\theta$ between $\vec{u} = 4\, \vec{i} - 3\, \vec{j}$ and $\vec{v} = 2\, \vec{i} + 5\, \vec{j}$.

6. Parallel vectors

Two vectors $\vec{v}$ and $\vec{w}$ are said to be parallel if there exists a nonzero scalar $\alpha$ so that $\vec{v} = \alpha \vec{w}$. In this case, the angle $\theta$ between $\vec{v}$ and $\vec{w}$ is 0 or $\pi$.

7. Example: Show vectors $\vec{v} = 3\, \vec{i} - \vec{j}$ and $\vec{w} = 6\, \vec{i} - 2\, \vec{j}$ are parallel.
8. Orthogonal vectors

If the angle $\theta$ between two nonzero vectors $\vec{v}$ and $\vec{w}$ is $\frac{\pi}{2}$, the vectors $\vec{v}$ and $\vec{w}$ are called orthogonal.

Theorem: Two vectors $\vec{v}$ and $\vec{w}$ are orthogonal if and only if

$$\vec{v} \cdot \vec{w} = 0$$

9. Example: Show vectors $\vec{v} = 2\vec{i} - \vec{j}$ and $\vec{w} = 3\vec{i} + 6\vec{j}$ are orthogonal.

10. Projection of a Vector onto Another Vector
Theorem: If \( \vec{v} \) and \( \vec{w} \) are two nonzero vectors, the projection of \( \vec{v} \) onto \( \vec{w} \) is

\[
\vec{v}_1 = \frac{\vec{v} \cdot \vec{w}}{\|\vec{w}\|^2} \vec{w}
\]

Theorem: The decomposition of \( \vec{v} \) into \( \vec{v}_1 \) and \( \vec{v}_2 \), where \( \vec{v}_1 \) is parallel to \( \vec{w} \) and \( \vec{v}_2 \) is perpendicular to \( \vec{w} \), is

\[
\vec{v}_1 = \frac{\vec{v} \cdot \vec{w}}{\|\vec{w}\|^2} \vec{w} \quad \vec{v}_2 = \vec{v} - \vec{v}_1
\]

11: Example: Find the vector projection of \( \vec{v} = \vec{i} + 3\vec{j} \) onto \( \vec{w} = \vec{i} + \vec{j} \).
5.6 Vectors in space

1. Rectangular coordinates in space

2. Distance formula in space

If \( P_1 = (x_1, y_1, z_1) \) and \( P_2 = (x_2, y_2, z_2) \) are two points in space, the distance \( d \) from \( P_1 \) to \( P_2 \) is

\[
d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.
\]

3. Example: Find the distance from \( P_1 = (-1, 3, 2) \) to \( P_2 = (4, -2, 5) \).

4. Representation of position vectors

A vector whose initial point is at the origin is called a position vector.

\[
\overrightarrow{v} = a \overrightarrow{i} + b \overrightarrow{j} + c \overrightarrow{k}.
\]

The scalars \( a, b, \) and \( c \) are called the components of the vector \( \overrightarrow{v} = a \overrightarrow{i} + b \overrightarrow{j} + c \overrightarrow{k} \).
5. Representation of any vectors

Suppose that \( \vec{v} \) is a vector with initial point \( P_1 = (x_1, y_1, z_1) \), not necessarily the origin, and the terminal point \( P_2 = (x_2, y_2, z_2) \). If \( \vec{v} = P_1P_2 \), then \( \vec{v} \) is equal to the position vector

\[
\vec{v} = (x_2 - x_1) \vec{i} + (y_2 - y_1) \vec{j} + (z_2 - z_1) \vec{k}.
\]

6. Example: Find the position vector of the vector \( \vec{v} = P_1P_2 \) if \( P_1 = (-1, 2, 3) \) and \( P_2 = (4, 6, 2) \).

7. Properties of vectors in terms of components.

Let \( \vec{v} = a_1 \vec{i} + b_1 \vec{j} + c_1 \vec{k} \) and \( \vec{w} = a_2 \vec{i} + b_2 \vec{j} + c_2 \vec{k} \) be two vectors, and let \( \alpha \) be a scalar. Then

\[
\begin{align*}
(1) \quad \vec{v} &= \vec{w} \quad \text{if and only if} \quad a_1 = a_2, b_1 = b_2, \text{ and } c_1 = c_2. \\
(2) \quad \vec{v} + \vec{w} &= (a_1 + a_2) \vec{i} + (b_1 + b_2) \vec{j} + (c_1 + c_2) \vec{k}. \\
(3) \quad \vec{v} - \vec{w} &= (a_1 - a_2) \vec{i} + (b_1 - b_2) \vec{j} + (c_1 - c_2) \vec{k}. \\
(4) \quad \alpha \vec{v} &= (\alpha a_1) \vec{i} + (\alpha b_1) \vec{j} + (\alpha c_1) \vec{k}. \\
(5) \quad \|\vec{v}\| &= \sqrt{a_1^2 + b_1^2 + c_1^2}.
\end{align*}
\]

8. Example: If \( \vec{v} = 2 \vec{i} + 3 \vec{j} - 2 \vec{k} \) and \( \vec{w} = 3 \vec{i} - 4 \vec{j} + 5 \vec{k} \), find:

\[
\begin{align*}
(1) \quad \vec{v} + \vec{w} & \quad (2) \quad \vec{v} - \vec{w} & \quad (3) \quad 3\vec{v} & \quad (4) \quad 2\vec{v} - 3\vec{w} & \quad (5) \quad \|\vec{v}\| \\
\end{align*}
\]
9. Unit vector in the direction of $\vec{v}$

For any nonzero vector $\vec{v}$, the vector

$$\vec{u} = \frac{\vec{v}}{\|\vec{v}\|}$$

is a unit vector that has the same direction as $\vec{v}$.

10. Example: Find a unit vector in the same direction as $\vec{v} = 2\vec{i} - 3\vec{j} - 6\vec{k}$.

11. Dot product

If $\vec{v} = a_1\vec{i} + b_1\vec{j} + c_1\vec{k}$ and $\vec{w} = a_2\vec{i} + b_2\vec{j} + c_2\vec{k}$ are two vectors, the dot product $\vec{v} \cdot \vec{w}$ is defined as

$$\vec{v} \cdot \vec{w} = a_1a_2 + b_1b_2 + c_1c_2.$$

12. Example: $\vec{v} = 2\vec{i} - 3\vec{j} + 6\vec{k}$ and $\vec{w} = 5\vec{i} + 3\vec{j} - \vec{k}$, find

(1) $\vec{v} \cdot \vec{w}$  (2) $\vec{w} \cdot \vec{v}$  (3) $\vec{v} \cdot \vec{v}$  (4) $\|\vec{v}\|$ 

12. Properties of dot product

If $\vec{u}$, $\vec{v}$, and $\vec{w}$ are vectors, then

(1) $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$
(2) $\vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}$
(3) $\vec{v} \cdot \vec{v} = \|\vec{v}\|^2$
(4) $\vec{0} \cdot \vec{v} = 0$

30
13. Angle between vectors

If $\vec{u}$ and $\vec{v}$ are two nonzero vectors, the angle $\theta$, $0 \leq \theta \leq \pi$, between $\vec{u}$ and $\vec{v}$ is determined by the formula

$$\cos \theta = \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\| \|\vec{v}\|}.$$ 

14. Example: Find the angle $\theta$ between $\vec{v} = 2\vec{i} - 3\vec{j} + 6\vec{k}$ and $\vec{w} = 2\vec{i} + 5\vec{j} - \vec{k}$.
5.7 The cross product

1. If \( \vec{v} = a_1 \vec{i} + b_1 \vec{j} + c_1 \vec{k} \) and \( \vec{w} = a_2 \vec{i} + b_2 \vec{j} + c_2 \vec{k} \) are two vectors, the cross product \( \vec{v} \times \vec{w} \) is defined as the vector

\[
\vec{v} \times \vec{w} = (b_1 c_2 - b_2 c_1) \vec{i} + (a_1 c_2 - a_2 c_1) \vec{j} + (a_1 b_2 - a_2 b_1) \vec{k}.
\]

Note that the cross product \( \vec{v} \times \vec{w} \) of two vectors is a vector.

2. Example: If \( \vec{v} = 2 \vec{i} + 3 \vec{j} + 5 \vec{k} \) and \( \vec{w} = \vec{i} + 2 \vec{j} + 3 \vec{k} \), find \( \vec{v} \times \vec{w} \).

3. Determinant

4. Example:

5. Cross product in terms of determinants
6. Example: If $\vec{v} = 2\vec{i} + 3\vec{j} + 5\vec{k}$ and $\vec{w} = \vec{i} + 2\vec{j} + 3\vec{k}$, find:

(1) $\vec{v} \times \vec{w}$  (2) $\vec{w} \times \vec{v}$  (3) $\vec{v} \times \vec{v}$  (4) $\vec{w} \times \vec{w}$

7. Algebraic properties of the cross product

If $\vec{u}$, $\vec{v}$, and $\vec{w}$ are vectors and $\alpha$ is a scalar, the

(1) $\vec{u} \times \vec{u} = \vec{0}$
(2) $\vec{u} \times \vec{v} = -\vec{v} \times \vec{u}$
(3) $\alpha(\vec{v} \times \vec{w}) = (\alpha \vec{v}) \times \vec{w} = \vec{v} \times (\alpha \vec{w})$
(4) $\vec{u} \times (\vec{v} + \vec{w}) = (\vec{u} \times \vec{v}) + (\vec{u} \times \vec{w})$

8. Geometric properties of the cross product

Let $\vec{u}$ and $\vec{v}$ be vectors in space.

(1) $\vec{u} \times \vec{v}$ is orthogonal to both $\vec{u}$ and $\vec{v}$.
(2) $\|\vec{u} \times \vec{v}\| = \|\vec{u}\|\|\vec{v}\| \sin \theta$, where $\theta$ is the angle between $\vec{u}$ and $\vec{v}$.
(3) $\|\vec{u} \times \vec{v}\|$ is the area of the parallelogram having $\vec{u} \neq \vec{0}$ and $\vec{v} \neq \vec{0}$ as adjacent sides.
(4) $\vec{u} \times \vec{v} = \vec{0}$ if and only if $\vec{u}$ and $\vec{v}$ are parallel.
9. Example: Find a vector that is orthogonal to $\vec{u} = 3\vec{i} - 2\vec{j} + \vec{k}$ and $\vec{v} = -\vec{i} + 3\vec{j} - \vec{k}$.

10. Example: Find the area of the parallelogram whose vertices are $P_1 = (0, 0, 0)$, $P_2 = (3, -2, 1)$, $P_3 = (-1, 3, -1)$, and $P_4 = (2, 1, 0)$.