Invariant polynomial subspaces

David Gómez-Ullate, Niky Kamran, Robert Milson

ACA 2007
Overview of some recent results

Motivation and background
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  - Key concept: exceptional subspace

Quasi-exact solvability beyond $\mathfrak{sl}_2$
Finite-gap potentials

Nonlinear models
- Key concept: deficiency
- Translation-invariant, quadratically non-linear operators
- Non-standard reduction of evolution equations (non-linear separation of variables)

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Basic setup

- Univariate differential operator: \( T_f[u] := f(x, u, u', \ldots, u^{(r)}) \) acting on \( \mathcal{P}_n(x) = \langle 1, x, \ldots, x^n \rangle \)
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Variations

- Linear models: look for \( \mathcal{M} \) preserved by “interesting” (e.g., rational coefficients) operators.
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Variations

- Linear models: look for $\mathcal{M}$ preserved by “interesting” (e.g., rational coefficients) operators.
- Non-linear models: look for “interesting” (e.g., low order, translation-invariant) operators preserving $\mathcal{P}_n$
**sl$_2$ approach to quasi-exact solvability**

**sl$_2$ generators**

\[
\begin{align*}
J^-_n &= D_x, \\
J^0_n &= xD_x - \frac{n}{2}, \\
J^+_n &= x^2D_x - nx, \\
n &= 0, 1, 2, 3, \ldots
\end{align*}
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**Theorem (Burnside)**

\[
\mathcal{D}_r(P_n) = \{ p(J_n^-, J_n^0, J_n^+) : \deg(p) = r \}
\]
**sl₂ approach to quasi-exact solvability**

### sl₂ generators

<table>
<thead>
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### Theorem (Burnside, G-U, K, M)

**Definition**

$D_r(M) := \{ T: \text{linear operator}, \text{ord}(T) \leq r, T(M) \subseteq M \}$

**Call** $M$ an $X_k$ (exceptional) subspace if

$D_2(M) \subset D_2(P_n)$

**Questions**

Is QES more general than the sl₂ algebraization?

Do exceptional subspaces exist?

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**Definition**

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\mathcal{D}_r(\mathcal{M}) := \{ T \text{ linear operator} : \text{ord}(T) \leq r \text{ and } T(\mathcal{M}) \subseteq \mathcal{M} \}
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- Is QES more general than the sl\(_2\) algebraization?
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- Is QES more general than the sl$_2$ algebraization?
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Quasi-exact solvability beyond sl\(_2\)

**Theorem (Post, Turbiner, 1995)**

*The codim. 1 subspace \( \mathcal{E}_n = \langle 1, x^2, x^3, \cdots, x^n \rangle \) is \( X_1 \).*
Quasi-exact solvability beyond $\text{sl}_2$

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*Proof.* $\mathcal{E}_n$ is preserved by $D_{xx} - \frac{2}{x} D_x$. 

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Every $X_1$ subspace $\mathcal{M}_n$ is projectively equivalent to $\mathcal{E}_n$. 
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**A QES potential with a non-$\text{sl}_2$ sector**

- Hamiltonian: $H = -D_{xx} + 2A^2 \cosh(2x) + 4An \cosh(x) - \frac{1}{2} \sech^2(x/2)$
The codim. 1 subspace \( \mathcal{E}_n = \langle 1, x^2, x^3, \ldots, x^n \rangle \) is \( X_1 \).

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  \[ \psi_{2k}(x) = \mu(x) p_k(z(x)), \quad p_k(z) \in \mathcal{P}_n(z), \quad k = 0, \ldots, n \]
  \[ T = \mu^{-1} H \mu, \quad \mu(x) = \exp(2A \cosh(x)) \text{sech}(x/2), \quad z = -\sinh^2(x/2) \]
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Quasi-exact solvability beyond sl$_2$  

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**Theorem (T,V, 1990)**

A complex-valued potential \( u(x) = \sum_{j=1}^{4} d_j \varphi(x - \omega_j) \), where \( \omega_1 = 0, \omega_4 = \omega_2 + \omega_3 \), and \( \omega_2, \omega_3 \) are the fundamental half-periods, is a solution of the stationary KdV hierarchy iff \( d_j = \ell_j (\ell_j + 1) \) for some \( \ell_j \in \mathbb{Z} \).
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**Associated Lamé potentials, QES Heun equations**

\[
H = -D_{xx} + \sum_{j=1}^{4} \ell_j(\ell_j + 1)\wp(x - \omega_j), \quad z = \wp(x), \quad w^2 = p(z)
\]

\[
= p(z)D_{zz} + \frac{1}{2}p'(z)D_z + \ell_1(\ell_1 + 1)z + \sum_{i=2}^{4} \ell_j(\ell_j + 1)\frac{p'(z_j)}{z - z_j}
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- $\ell_1 > \ell_2 \in \mathbb{Z}_+, \ell_3, \ell_4 \in \{0, 1\}, \omega_1, i\omega_2 \in \mathbb{R} \Rightarrow T$ is a finite-gap potential

  4 algebraic sectors: $\mathcal{A} = \bigoplus_{i=0}^{3} \mu_i(x)\mathcal{P}_{n_i}(z)$ with $\dim \mathcal{A} = \sum_i n_i = 2\ell_1 + 1$

  Band edges, spectral curve: $\nu^2 = \det(E - [H|\mathcal{A}]) = \prod_{j=0}^{2\ell_1}(E - E_j)$. 
A complex-valued potential \( u(x) = \sum_{j=1}^{4} d_j \phi(x - \omega_j) \), where \( \omega_1 = 0, \omega_4 = \omega_2 + \omega_3 \), and \( \omega_2, \omega_3 \) are the fundamental half-periods, is a solution of the stationary KdV hierarchy iff \( d_j = \ell_j(\ell_j + 1) \) for some \( \ell_j \in \mathbb{Z} \).

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Band edges, spectral curve: \( \nu^2 = \det(E - [H|\mathcal{A}]) = \prod_{j=0}^{2\ell_1}(E - E_j) \).
An X\(_1\) elliptic, finite-gap potential

**Theorem (Gesztesy, Weikard)**

An elliptic potential \(u(x)\) is a solution of the stationary KdV hierarchy iff

\[
    u(x) = \sum_{j=1}^{M} \ell_j (\ell_j + 1) \wp(x - \omega_j), \quad \ell_j \in \mathbb{Z},
\]

where

\[
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An $X_1$ elliptic, finite-gap potential

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Non-linear separation of variables

**Problem**

Fix $n$ and consider $\mathcal{P}_n(x)$. Find all translation-invariant differential operators $T_Q[u] := Q(u_0, u_1, \ldots, u_n)$, where $Q$ is a quadratic polynomial such that $T_Q(\mathcal{P}_n) \subset \mathcal{P}_n$. 

Application

Consider the autonomous evolution equation $u_t = Q[u]$ where $T_Q(p(x)) = \sum_{i=0}^{n} Q_i(C_0, \ldots, C_n) x^i$, $p(x) = C_0 + \cdots + C_n x^n$.

Use the solution ansatz $u(t, x) = \sum_{i=0}^{n} C_i(t) x^i$. The PDE reduces to the ODE $C_i'(t) = Q_i(C_0, \ldots, C_n)$. 


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Invariant polynomial subspaces
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Fix $n$ and consider $\mathcal{P}_n(x)$. Let $T_Q[u] := Q(x, u_0, u_1, \ldots, u_n)$ be a differential operator, polynomial in $x$ and derivatives $u_j = u^{(j)}(x)$. Say that $T$ has deficiency $m$ if $T(\mathcal{P}_n) \subset \mathcal{P}_{n-m}$ and $T(\mathcal{P}_n) \not\subset \mathcal{P}_{n-m-1}$. Say that monomial deficiency is $k$ if each term has deficiency $k$. 

Observation: $m \geq k$. Consider $T[u] = 3u_2^2 - 4u_0u_2$ acting on $\mathcal{P}_4$. Monomial deficiency is $-2$. However, $T(\mathcal{P}_4) \subset \mathcal{P}_4$ (leading terms annihilated). Therefore, actual deficiency is 0.
Definition

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Algebra of differential polynomials

Gradings: $\ell =$ deg. nonlinearity, $k =$ mon. deficiency, $m =$ deficiency

$\mathcal{T} = \mathbb{R}[x, u_0, u_1, \ldots, u_n]$ \quad (n is fixed)
Algebra of differential polynomials

Gradings: $\ell =$ deg. nonlinearity, $k =$ mon. deficiency, $m =$ deficiency

- $\mathcal{I} = \mathbb{R}[x, u_0, u_1, \ldots, u_n]$ (n is fixed)
  $$= \bigoplus_{\ell=0}^{\infty} \mathcal{I}_\ell$$

- $\mathcal{I}_\ell = \text{span}\{x^j u_{i_1} u_{i_2} \cdots u_{i_\ell} \mid 0 \leq j < \infty\}.$
## Algebra of differential polynomials

Gradings: \( \ell = \) deg. nonlinearity, \( k = \) mon. deficiency, \( m = \) deficiency

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  \]
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- \( \mathcal{T}_{\ell,k} = \text{span}\{x^j u_{i_1} u_{i_2} \cdots u_{i_\ell} \mid \text{wt} = k\} \), \( \text{wt}(u_i) = n - i \), \( \text{wt}(x) = 1 \).
Gradings: $\ell = \text{deg. nonlinearity}$, $k = \text{mon. deficiency}$, $m = \text{deficiency}$

- $T = \mathbb{R}[x, u_0, u_1, \ldots, u_n]$ (n is fixed)
  
  $= \bigoplus_{\ell=0}^{\infty} T_{\ell} = \bigoplus_{\ell=0}^{\infty} T_{\ell,k} = \bigoplus_{\ell=0}^{\infty} \bigoplus_{m=0}^{k} T_{\ell,k,m}$

- $T_{\ell} = \text{span}\{x^j u_{i_1} u_{i_2} \cdots u_{i_{\ell}} \mid 0 \leq j < \infty\}$.

- $T_{l,k} = \text{span}\{x^j u_{i_1} u_{i_2} \cdots u_{i_{\ell}} \mid \text{wt}(x) = k\}$, $\text{wt}(u_i) = n - i$, $\text{wt}(x) = 1$.

- $T_{l,k,m} = \text{span}\{x^m v_{i_1} \cdots v_{i_{\ell}} \mid \text{wt}(x) = k\}$ where

  $v_j = \sum_{i=0}^{n-j} (-1)^i \frac{x^i}{i!} u_{i+j}$,  \quad $u_j = \sum_{i=0}^{n-j} \frac{x^i}{i!} v_{i+j}$,  \quad $j = 0, \ldots, n$

  $\text{wt}(v_j) = n - j$  \quad $v_j : x^i \mapsto \delta^i_j$  \quad (Maximal deficiency)
Algebra of differential polynomials

Gradings: $\ell = \text{deg. nonlinearity}, k = \text{mon. deficiency}, m = \text{deficiency}$

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Algebra of differential polynomials

Gradings: $\ell = \deg \text{ nonlinearity, } k=\text{mon. deficiency, } m=\text{deficiency}$

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- $\mathcal{T} = \bigoplus_{\ell=0}^{\infty} \mathcal{T}_\ell = \bigoplus_{\ell,k=0}^{\infty} \mathcal{T}_{\ell,k} = \bigoplus_{\ell,k=0}^{\infty} \bigoplus_{m=0}^{k} \mathcal{T}_{\ell,k,m}$
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- $\mathcal{T}_{l,k} = \text{span}\{x^j u_{i_1} u_{i_2} \cdots u_{i_\ell} \mid \text{wt} = k\}$, $\text{wt}(u_i) = n - i$, $\text{wt}(x) = 1$.
- $\mathcal{T}_{l,k,m} = \text{span}\{x^m v_{i_1} \cdots v_{i_\ell} \mid \text{wt} = k\}$ where
  \[
  v_j = \sum_{i=0}^{n-j} (-1)^i \frac{x^i}{i!} u_{i+j}, \quad u_j = \sum_{i=0}^{n-j} \frac{x^i}{i!} v_{i+j}, \quad j = 0, \ldots, n
  \]
  \[
  \text{wt}(v_j) = n - j \quad v_j : x^i \mapsto \delta_{ij} \quad (\text{Maximal deficiency})
  \]

**Proposition**

Let $T = P(x, u_0, \ldots, u_n) = Q(x, v_0, \ldots, v_n)$. Then,

\[
\text{deficiency } T = n - \text{deg}_x Q.
\]
Autonomous (translation-invariant) operators

**Definition**

\[ v_j = \sum_{i=0}^{n-j} (-1)^i \frac{x^i}{i!} u_{i+j}, \quad u_j = \sum_{i=0}^{n-j} \frac{x^i}{i!} v_{i+j}, \quad j = 0, \ldots, n \]

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Example: \( n = 4 \)

\[ v_4 = u_4 \]
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Proposition

The operators \( l_j \) are autonomous (no \( x \) variable) and maximal deficiency (turn all polynomials into constants). Let \( T = P(u_0, \ldots, u_n) = Q(x, v_0, \ldots, v_n) \)

Then \( T = Q(\xi, l_n, \ldots, l_2, 0, l_0) \) and deficiency \( T = n - \deg_{\xi} Q \).
Autonomous operators: an example

Problem

Fix $n = 4$ and consider $T = C_{02}u_0u_2 + C_{11}u_1^2$

$T$ is quadratically non-linear, $\ell = 2$, and has monomial deficiency $k = -2$.

Find conditions on $C_{02}, C_{11}$ s.t. deficiency $T \geq 0$, i.e. so that $T$ preserves $\mathcal{P}_4$. 
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Solution

Rewrite $T$ using $\xi, I_4, I_3, I_2, I_0$

$v_4 = u_4$
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### Solution

Rewrite $T$ using $\xi, l_4, l_3, l_2, l_0$

<table>
<thead>
<tr>
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</tr>
</thead>
<tbody>
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Conclusion: $T$ preserves $\mathcal{P}_4$ iff $T \propto 4u_0 u_2 - 3u_1^2$
Autonomous, quadratically non-linear operators

- Fix $n$. Set $\mathcal{Q} := \bigoplus_{k=-n}^{n} \mathcal{Q}_k$, $\mathcal{Q}_k := \langle u_i u_j : i + j = n + k, \ 0 \leq i, j \leq n \rangle$. 
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<table>
<thead>
<tr>
<th>$k$</th>
<th>$Q_{k,r}$</th>
<th>$m$</th>
</tr>
</thead>
<tbody>
<tr>
<td>-4</td>
<td>$u_0^2$</td>
<td>-4</td>
</tr>
<tr>
<td>-3</td>
<td>$u_0 u_1$</td>
<td>-3</td>
</tr>
<tr>
<td>-2</td>
<td>$u_1^2$</td>
<td>-2</td>
</tr>
<tr>
<td>1</td>
<td>$3u_1^2 - 4u_0 u_2$</td>
<td>-1</td>
</tr>
<tr>
<td>0</td>
<td>$u_1 u_2$</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>$2u_2^2 - 3u_1 u_3$</td>
<td>2</td>
</tr>
<tr>
<td>1</td>
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<td>1</td>
</tr>
<tr>
<td>2</td>
<td>$u_2^2 - 3u_1 u_4$</td>
<td>3</td>
</tr>
<tr>
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Quadratic autonomous operators acting on $\mathcal{P}_4$
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Thank you for your attention.

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