Lie Superalgebras of Matrix Differential Operators

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What is a Lie superalgebra?

A \((\mathbb{Z}_2-)\)graded vector space \(g = g_0 \oplus g_1\), equipped with a graded bracket \(\langle \ , \ \rangle\), satisfying graded skew-symmetry,

\[
\langle A, B \rangle = -(-1)^{ab} \langle B, A \rangle ,
\]

and the graded Jacobi identity,

\[
\langle \langle A, B \rangle , C \rangle = \langle A, \langle B, C \rangle \rangle - (-1)^{ab} \langle B, \langle A, C \rangle \rangle ,
\]

where \(A \in g_a, B \in g_b, C \in g_c\).

Symmetrical form of the graded Jacobi identity:

\[
(-1)^{ac} \langle A, \langle B, C \rangle \rangle + \text{cyclic} = 0.
\]
Some facts about Lie superalgebras:

- The even subspace $\mathfrak{g}_0$ is a Lie algebra.

- The odd subspace $\mathfrak{g}_1$ generates a Lie superalgebra contained in $\mathfrak{g}$.

- Every $\mathbb{Z}_2$-graded associative algebra has a Lie superalgebra structure, with

  $\langle A, B \rangle = A \cdot B - (-1)^{ab} B \cdot A$, where $A \in \mathfrak{g}_a$, $B \in \mathfrak{g}_b$.

- There is a wide variety of interesting classes of Lie superalgebras.

The Canonical Realization, \( \text{spl}(2, 1) \).

Basis for \( g_1 \) (odd operators):
\[
A_1 = \begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 0 \\ x\partial_x - \lambda & 0 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 & 0 \\ \partial_x & 0 \end{bmatrix}.
\]

Basis for \( g_0 \) (the even subalgebra):
\[
X = A_1A_2 + A_2A_1 = \begin{bmatrix} x^2\partial_x - \lambda x & 0 \\ 0 & x^2\partial_x - (\lambda - 1)x \end{bmatrix},
\]
\[
Y = B_1B_2 + B_2B_1 = \begin{bmatrix} \partial_x & 0 \\ 0 & \partial_x \end{bmatrix},
\]
\[
H_1 = A_1B_2 + B_2A_1, \quad H_2 = A_2B_1 + B_1A_2.
\]

The graded bracket:
\[
\langle A, B \rangle = AB - (-1)^{ab} BA, \text{ where } A \in g_a, \ B \in g_b.
\]
Properties of this Realization:

- The realization is a map into a Weyl-Clifford algebra $\mathcal{A}(m) \otimes \mathrm{Cl}(2n)$.
- The matrices are $2^n \times 2^n$ with coefficients from a Weyl algebra.
- The action of the even subalgebra is by 1st-order differential operators.
- The even subalgebra consists of diagonal MDO’s.
- A parameter $\lambda$ determines various f.d. representations.
- Others....
Why Clifford Algebras?

Poincaré-Birkhoff-Witt Theorem for Lie Superalgebras.

**Corollary to PBW:** If $V$ is a generic, irreducible representation of a Lie superalgebra $g = g_0 \oplus g_1$, then $V$ is a direct sum of $2^r$ irreducible representations of $g_0$, where $r \leq \dim(g_1)$.

$\Rightarrow g_0$ should act “diagonally” and $g_1$ should “shift” between $g_0$-modules.

$\Rightarrow$ We should expect $2^r \times 2^r$ MDO’s.
Motivation:

- Quasi-exactly solvable (QES) DE’s
- Representation theory/geometry of Lie superalgebras
- Representation theory/geometry of Lie algebras
- Compactness of information
Orthosymplectic Lie superalgebras $\text{osp}(m, 2n)$

The Lie superalgebra $\text{osp}(m, 2n)$ is the space of all matrices

$$\begin{bmatrix} A & B \\ JB^TQ & D \end{bmatrix},$$

where

$$AQ + QA^T = 0, JD + D^TJ = 0,$$

$$Q^T = Q, J^T = -J,$$

$$Q^2 = I_m, J^2 = -I_{2n},$$

$\text{osp}(m, 2n)_0 = \text{block diagonal matrices} = \mathfrak{so}(m) \oplus \mathfrak{sp}(n)$,

$\text{osp}(m, 2n)_1 = \text{the complement} = \mathbb{C}^m \otimes \mathbb{C}^{2n}$.
Realization of $\text{osp}(3, 2)$ by $4 \times 4$ MDO’s in one variable

Basis for $g_0$ (the even subalgebra):

\[
X_1 = x(2 + \mu - x\partial_x)II + x(HI - IH), \quad Y_1 = \partial_xII,
\]
\[
X_2 = XX, \quad Y_2 = YY,
\]
\[
H_1 = [X_1, Y_1], \quad H_2 = [X_2, Y_2].
\]

Basis for $g_1$ (odd operators):

\[
B_0 = IY + \partial_x^2 YH
\]
\[
B_1 = xIY - (2 + \mu - x\partial_x)\partial_x YH + \partial_x YI
\]
\[
B_2 = x^2 IY + \left(\frac{5}{2} + \mu - x\partial_x\right)^2 \frac{3}{4} YH - 2 \left(\mu + \frac{5}{2} - x\partial_x\right) YI
\]
\[
A_0 = XH - \partial_x^2 IX
\]
\[
A_1 = xXH + (2 + \mu - x\partial_x)\partial_x IX + \partial_x HX
\]
\[
A_2 = x^2 XH - \left(\frac{5}{2} + \mu - x\partial_x\right)^2 \frac{3}{4} IX - 2 \left(\mu + \frac{5}{2} - x\partial_x\right) HX
\]
Notation:

\{II = I \otimes I, HI = H \otimes I, XI = X \otimes I, \ldots\} \text{ is a basis for the 16-dimensional Clifford algebra of } 4 \times 4 \text{ matrices,}

\{I = E_{11} + E_{22}, H = E_{11} - E_{22}, X = E_{12}, Y = E_{21}\} \text{ is a basis for the 4-dimensional Clifford algebra of } 2 \times 2 \text{ matrices.}

“\otimes” \text{ is Kronecker’s tensor product of matrices.}
Roots and Weights:

Root Vectors of \( \text{osp}(3, 2) \)  

Weights of a Representation, \( \mu = 0 \)
Realization of \( \mathfrak{osp}(3, 2) \) by \( 8 \times 8 \) MDO’s in two variables

The even subalgebra:

\[
\begin{align*}
X_1 & = (\mu x - x^2 \partial_x)III + x(2III + HII - IIH), \\
H_1 & = (2x \partial_x - \mu)III - (2III + HII - IIH), \\
Y_1 & = pIII, \\
X_2 & = (\lambda y - y^2 \partial_y)III + \frac{1}{2}y(3III + IIH + IHI + HII), \\
H_2 & = (2y \partial_y - \lambda)III - \frac{1}{2}(3III + IIH + IHI + HII), \\
Y_2 & = qIII,
\end{align*}
\]

The odd operators:

\[
\begin{align*}
B_0 & = (a_1XII + a_2XIH + a_3XHI + a_4XHH) \\
& + \partial_x (b_1IXI + b_2IXH + b_3HXI + b_4HXH + b_5XYX) \\
& + \partial^2_x (c_1IX + c_2IXH + c_3HIX + c_4HHX) \\
& + \partial_y (d_1IY + d_2IHY + d_3HIX + d_4HHY) \\
& + \partial_x \partial_y (e_1IYI + e_2IYH + e_3HYI + e_4HYH + e_5XYX) \\
& + \partial^2_x \partial_y (f_1YII + f_2YIH + f_3YHI + f_4YHH),
\end{align*}
\]

\[B_1 = [X_1, B_0], \ B_2 = [X_1, B_1], \ A_0 = [X_2, B_0], \ A_1 = [X_2, B_1], \ A_2 = [X_2, B_2].\]
Undetermined coefficients:

\[ a_1 =? \]
\[ a_2 =? \]
\[ \ldots \]
\[ f_4 =? \]

Let’s look at the Maple worksheet....
Future work:

- Obtain realizations for other classes of Lie superalgebras.
- Ground the search in more theory.
- Check the realizations (using “real” mathematics).
- Express realizations more efficiently.
- Explore equivalence of realizations.
- Others....
References:

