3.6 The Real Zeros of a Polynomial Function

**PREPARING FOR THIS SECTION**
Before getting started, review the following:

- Classification of Numbers (Appendix, Section A.1, p. 952)
- Factoring Polynomials (Appendix, Section A.3, pp. 969-971)
- Synthetic Division (Appendix, Section A.4, pp. 979-982)
- Polynomial Division (Appendix, Section A.4, pp. 977-979)
- Quadratic Formula (Appendix, Section A.5, pp. 992-995)

Now work the ‘Are You Prepared?’ problems on page 230.

**OBJECTIVES**
1. Use the Remainder and Factor Theorems
2. Use the Rational Zeros Theorem
3. Find the Real Zeros of a Polynomial Function
4. Solve Polynomial Equations
5. Use the Theorem for Bounds on Zeros
6. Use the Intermediate Value Theorem

In this section, we discuss techniques that can be used to find the real zeros of a polynomial function. Recall that if \( r \) is a real zero of a polynomial function \( f \) then \( f(r) = 0 \), \( r \) is an \( x \)-intercept of the graph of \( f \), and \( r \) is a solution of the equation \( f(x) = 0 \). For polynomial and rational functions, we have seen the importance of the zeros for graphing. In most cases, however, the zeros of a polynomial function are difficult to find using algebraic methods. No nice formulas like the quadratic formula are available to help us find zeros for polynomials of degree 3 or higher. Formulas do exist for solving any third- or fourth-degree polynomial equation, but they are somewhat complicated. No general formulas exist for polynomial equations of degree 5 or higher. Refer to the Historical Feature at the end of this section for more information.

1. **Use the Remainder and Factor Theorems**

When we divide one polynomial (the dividend) by another (the divisor), we obtain a quotient polynomial and a remainder, the remainder being either the zero polynomial or a polynomial whose degree is less than the degree of the divisor. To check our work, we verify that

\[
(\text{Quotient})(\text{Divisor}) + \text{Remainder} = \text{Dividend}
\]

This checking routine is the basis for a famous theorem called the **division algorithm** for polynomials, which we now state without proof.

**Theorem**

**Division Algorithm for Polynomials**

If \( f(x) \) and \( g(x) \) denote polynomial functions and if \( g(x) \) is a polynomial whose degree is greater than zero, then there are unique polynomial functions \( q(x) \) and \( r(x) \) such that

\[
\frac{f(x)}{g(x)} = q(x) + \frac{r(x)}{g(x)} \quad \text{or} \quad f(x) = q(x)g(x) + r(x)
\]  \hspace{1cm} (1)

where \( r(x) \) is either the zero polynomial or a polynomial of degree less than that of \( g(x) \).

* A systematic process in which certain steps are repeated a finite number of times is called an **algorithm**. For example, long division is an algorithm.
In equation (1), $f(x)$ is the **dividend**, $g(x)$ is the **divisor**, $q(x)$ is the **quotient**, and $r(x)$ is the **remainder**.

If the divisor $g(x)$ is a first-degree polynomial of the form

$$g(x) = x - c, \quad c \text{ a real number}$$

then the remainder $r(x)$ is either the zero polynomial or a polynomial of degree 0. As a result, for such divisors, the remainder is some number, say $R$, and we may write

$$f(x) = (x - c)q(x) + R \quad (2)$$

This equation is an identity in $x$ and is true for all real numbers $x$. Suppose that $x = c$. Then equation (2) becomes

$$f(c) = (c - c)q(c) + R$$

$$f(c) = R$$

Substitute $f(c)$ for $R$ in equation (2) to obtain

$$f(x) = (x - c)q(x) + f(c) \quad (3)$$

We have now proved the **Remainder Theorem**.

**Remainder Theorem**

Let $f$ be a polynomial function. If $f(x)$ is divided by $x - c$, then the remainder is $f(c)$.

**Example 1**

**Using the Remainder Theorem**

Find the remainder if $f(x) = x^3 - 4x^2 - 5$ is divided by

(a) $x - 3$ \hspace{1cm} (b) $x + 2$

**Solution**

(a) We could use long division or synthetic division, but it is easier to use the Remainder Theorem, which says that the remainder is $f(3)$.

$$f(3) = (3)^3 - 4(3)^2 - 5 = 27 - 36 - 5 = -14$$

The remainder is $-14$.

(b) To find the remainder when $f(x)$ is divided by $x + 2 = x - (-2)$, we evaluate $f(-2)$.

$$f(-2) = (-2)^3 - 4(-2)^2 - 5 = -8 - 16 - 5 = -29$$

The remainder is $-29$.

Compare the method used in Example 1(a) with the method used in Example 4 on page 981 (synthetic division). Which method do you prefer? Give reasons.

**Note** A graphing utility provides another way to find the value of a function, using the eVALUEate feature. Consult your manual for details. See Figure 72 for the results of Example 1(a).

**Factor Theorem**

Let $f$ be a polynomial function. Then $x - c$ is a factor of $f(x)$ if and only if $f(c) = 0$. 

The Factor Theorem actually consists of two separate statements:

1. If \( f(c) = 0 \), then \( x - c \) is a factor of \( f(x) \).
2. If \( x - c \) is a factor of \( f(x) \), then \( f(c) = 0 \).

The proof requires two parts.

**Proof**

1. Suppose that \( f(c) = 0 \). Then, by equation (3), we have
   \[
   f(x) = (x - c)q(x) + f(c)
   = (x - c)q(x) + 0
   = (x - c)q(x)
   \]
   for some polynomial \( q(x) \). That is, \( x - c \) is a factor of \( f(x) \).
2. Suppose that \( x - c \) is a factor of \( f(x) \). Then there is a polynomial function \( q \) such that
   \[
   f(x) = (x - c)q(x)
   \]
   Replacing \( x \) by \( c \), we find that
   \[
   f(c) = (c - c)q(c) = 0 \cdot q(c) = 0
   \]
   This completes the proof.

One use of the Factor Theorem is to determine whether a polynomial has a particular factor.

**EXAMPLE 2 Using the Factor Theorem**

Use the Factor Theorem to determine whether the function

\[
f(x) = 2x^3 - x^2 + 2x - 3
\]

has the factor

(a) \( x - 1 \)  
(b) \( x + 2 \)

**Solution**

The Factor Theorem states that if \( f(c) = 0 \) then \( x - c \) is a factor.

(a) Because \( x - 1 \) is of the form \( x - c \) with \( c = 1 \), we find the value of \( f(1) \). We choose to use substitution.

\[
f(1) = 2(1)^3 - (1)^2 + 2(1) - 3 = 2 - 1 + 2 - 3 = 0
\]

See also Figure 73(a). By the Factor Theorem, \( x - 1 \) is a factor of \( f(x) \).

(b) We first need to write \( x + 2 \) in the form \( x - c \). Since \( x + 2 = x - (-2) \), we find the value of \( f(-2) \). See Figure 73(b). Because \( f(-2) = -27 \neq 0 \), we conclude from the Factor Theorem that \( x - (-2) = x + 2 \) is not a factor of \( f(x) \).
In Example 2, we found that $x - 1$ was a factor of $f$. To write $f$ in factored form, we can use long division or synthetic division. Using synthetic division, we find that

$$
\begin{array}{c|cc}
1 & -1 & 2 & -3 \\
 & & 2 & 1 & 3 \\
\hline
 & 2 & 1 & 3 \\
\end{array}
$$

The quotient is $q(x) = 2x^2 + x + 3$ with a remainder of 0, as expected. We can write $f$ in factored form as

$$f(x) = 2x^3 - x^2 + 2x - 3 = (x - 1)(2x^2 + x + 3)$$

**NOW WORK PROBLEM 11.**

The next theorem concerns the number of real zeros that a polynomial function may have. In counting the zeros of a polynomial, we count each zero as many times as its multiplicity.

**Theorem**

**Number of Real Zeros**

A polynomial function of degree $n$, $n \geq 1$, has at most $n$ real zeros.

**Proof** The proof is based on the Factor Theorem. If $r$ is a zero of a polynomial function $f$, then $f(r) = 0$ and, hence, $x - r$ is a factor of $f(x)$. Each zero corresponds to a factor of degree 1. Because $f$ cannot have more first-degree factors than its degree, the result follows.

2 **Use the Rational Zeros Theorem**

The next result, called the **Rational Zeros Theorem**, provides information about the rational zeros of a polynomial with integer coefficients.

**Theorem**

**Rational Zeros Theorem**

Let $f$ be a polynomial function of degree 1 or higher of the form

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0, \quad a_n \neq 0, a_0 \neq 0$$

where each coefficient is an integer. If $\frac{p}{q}$, in lowest terms, is a rational zero of $f$, then $p$ must be a factor of $a_0$, and $q$ must be a factor of $a_n$.

**EXAMPLE 3**

**Listing Potential Rational Zeros**

List the potential rational zeros of

$$f(x) = 2x^3 + 11x^2 - 7x - 6$$

Because $f$ has integer coefficients, we may use the **Rational Zeros Theorem**. First, we list all the integers $p$ that are factors of $a_0 = -6$ and all the integers $q$ that are factors of the leading coefficient $a_3 = 2$.

- $p$: $\pm 1, \pm 2, \pm 3, \pm 6$
- $q$: $\pm 1, \pm 2$
Now we form all possible ratios \( \frac{p}{q} \).

\[
\frac{p}{q} : \pm 1, \pm 2, \pm 3, \pm \frac{1}{2}, \pm \frac{3}{2}
\]

If \( f \) has a rational zero, it will be found in this list, which contains 12 possibilities.

NOW WORK PROBLEM 21.

Be sure that you understand what the Rational Zeros Theorem says: For a polynomial with integer coefficients, if there is a rational zero, it is one of those listed. The Rational Zeros Theorem does not say that if a rational number is in the list of potential rational zeros, then it is a zero. It may be the case that the function does not have any rational zeros.

The Rational Zeros Theorem provides a list of potential rational zeros of a function \( f \). If we graph \( f \), we can get a better sense of the location of the \( x \)-intercepts and test to see if they are rational. We can also use the potential rational zeros to select our initial viewing window to graph \( f \) and then adjust the window based on the results. The graphs shown throughout the text will be those obtained after setting the final viewing window.

## Example 4

**Finding the Real Zeros of a Polynomial Function**

Continue working with Example 3 to find the rational zeros of

\[
f(x) = 2x^3 + 11x^2 - 7x - 6
\]

**Solution**

We gather all the information that we can about the zeros.

**Step 1:** Since \( f \) is a polynomial of degree 3, there are at most three real zeros.

**Step 2:** We list the potential rational zeros obtained in Example 3:

\[
\pm 1, \pm 2, \pm 3, \pm \frac{1}{2}, \pm \frac{3}{2}.
\]

**Step 3:** We could, of course, use the Factor Theorem to test each potential rational zero to see if the value of \( f \) is zero. This is not very efficient. The graph of \( f \) will tell us approximately where the real zeros are. So we only need to test those rational zeros that are nearby. Figure 74 shows the graph of \( f \). We see that \( f \) has three zeros: one near \(-6\), one between \(-1\) and \(0\), and one near \(1\). From our original list of potential rational zeros, we will test \( -6 \), using the Factor Theorem.

**Step 4:** Because \( f(-6) = 0 \), we know that \(-6\) is a zero and \( x + 6 \) is a factor of \( f \).

We can use long division or synthetic division to factor \( f \).

\[
f(x) = 2x^3 + 11x^2 - 7x - 6 = (x + 6)(2x^2 - x - 1)
\]

Now any solution of the equation \( 2x^2 - x - 1 = 0 \) will be a zero of \( f \). Because of this, we call the equation \( 2x^2 - x - 1 = 0 \) a **depressed equation** of \( f \). Since the degree of the depressed equation of \( f \) is less than that of the original polynomial, we work with the depressed equation to find the zeros of \( f \).
The depressed equation $2x^2 - x - 1 = 0$ is a quadratic equation with discriminant $b^2 - 4ac = (-1)^2 - 4(2)(-1) = 9 > 0$. The equation has two real solutions, which can be found by factoring.

$$2x^2 - x - 1 = (2x + 1)(x - 1) = 0$$
$$2x + 1 = 0 \quad \text{or} \quad x - 1 = 0$$
$$x = -\frac{1}{2} \quad \text{or} \quad x = 1$$

The zeros of $f$ are $-6$, $-\frac{1}{2}$, and $1$. Because $f(x) = (x + 6)(2x^2 - x - 1)$, we completely factor $f$ as follows:

$$f(x) = 2x^3 + 11x^2 - 7x - 6 = (x + 6)(2x^2 - x - 1) = (x + 6)(2x + 1)(x - 1)$$

Notice that the three zeros of $f$ are among those given in the list of potential rational zeros in Example 3.

### Steps for Finding the Real Zeros of a Polynomial Function

**Step 1:** Use the degree of the polynomial to determine the maximum number of zeros.

**Step 2:** If the polynomial has integer coefficients, use the Rational Zeros Theorem to identify those rational numbers that potentially can be zeros.

**Step 3:** Using a graphing utility, graph the polynomial function.

**Step 4:**

(a) Use EVALUEate, substitution, synthetic division, or long division to test a potential rational zero based on the graph.

(b) Each time that a zero (and thus a factor) is found, repeat Step 4 on the depressed equation. In attempting to find the zeros, remember to use (if possible) the factoring techniques that you already know (special products, factoring by grouping, and so on).

### Example 5

**Finding the Real Zeros of a Polynomial Function**

Find the real zeros of $f(x) = x^5 - x^4 - 4x^3 + 8x^2 - 32x + 48$. Write $f$ in factored form.

**Solution**

**Step 1:** There are at most five real zeros.

**Step 2:** To obtain the list of potential rational zeros, we write the factors $p$ of $a_0 = 48$ and the factors $q$ of the leading coefficient $a_5 = 1$.

$$p: \pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 8, \pm 12, \pm 16, \pm 24, \pm 48$$

$$q: \pm 1$$

The potential rational zeros consist of all possible quotients $\frac{p}{q}$:

$$\frac{p}{q}: \pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 8, \pm 12, \pm 16, \pm 24, \pm 48$$
**Figure 75**

STEP 3: Figure 75 shows the graph of $f$. The graph has the characteristics that we expect of the given polynomial of degree 5: no more than four turning points, $y$-intercept 48, and it behaves like $y = x^5$ for large $|x|$.

STEP 4: Since $-3$ appears to be a zero and $-3$ is a potential rational zero, we evaluate $f$ at $-3$ and find that $f(-3) = 0$. We use synthetic division to factor $f$.

$$
\begin{array}{c|ccccc}
-3 & 1 & -4 & 8 & -32 & 48 \\
   &   & 12 & -24 & 48 & -48 \\
---&---&---&---&---&---
1 & -4 & 8 & -16 & 16 & 0 \\
\end{array}
$$

We can factor $f$ as

$$f(x) = x^5 - x^4 - 4x^3 + 8x^2 - 32x + 48 = (x + 3)(x^4 - 4x^3 + 8x^2 - 16x + 16)$$

We now work with the first depressed equation:

$q_1(x) = x^4 - 4x^3 + 8x^2 - 16x + 16 = 0$

Repeat Step 4: In looking back at Figure 75, it appears that 2 might be a zero of even multiplicity. We check the potential rational zero 2 and find that $f(2) = 0$. Using synthetic division,

$$
\begin{array}{c|ccccccc}
2 & 1 & -4 & 8 & -16 & 16 \\
   &   & 2 & -4 & 8 & -16 \\
---&---&---&---&---&---
1 & 2 & -4 & 8 & -16 & 0 \\
\end{array}
$$

Now we can factor $f$ as

$$f(x) = (x + 3)(x - 2)(x^3 - 2x^2 + 4x - 8)$$

Repeat Step 4: The depressed equation $q_2(x) = x^3 - 2x^2 + 4x - 8 = 0$ can be factored by grouping.

$$x^3 - 2x^2 + 4x - 8 = (x^3 - 2x^2) + (4x - 8) = x^2(x - 2) + 4(x - 2) = (x - 2)(x^2 + 4) = 0$$

$x - 2 = 0$ or $x^2 + 4 = 0$

$x = 2$

Since $x^2 + 4 = 0$ has no real solutions, the real zeros of $f$ are $-3$ and 2, with 2 being a zero of multiplicity 2. The factored form of $f$ is

$$f(x) = (x + 3)(x - 2)^2(x^2 + 4)$$

NOW WORK PROBLEM 39.

### 4 Solve Polynomial Equations

**EXAMPLE 6** **Solving a Polynomial Equation**

Solve the equation: $$x^5 - x^4 + 4x^3 + 8x^2 - 32x + 48 = 0$$

**Solution**

The solutions of this equation are the zeros of the polynomial function $f(x) = x^5 - x^4 + 4x^3 + 8x^2 - 32x + 48$

Using the result of Example 5, the real zeros are $-3$ and 2. These are the real solutions of the equation $x^5 - x^4 + 4x^3 + 8x^2 - 32x + 48 = 0$.

NOW WORK PROBLEM 63.
In Example 5, the quadratic factor \( x^2 + 4 \) that appears in the factored form of \( f(x) \) is called irreducible, because the polynomial \( x^2 + 4 \) cannot be factored over the real numbers. In general, we say that a quadratic factor \( ax^2 + bx + c \) is irreducible if it cannot be factored over the real numbers, that is, if it is prime over the real numbers.

Refer back to Examples 4 and 5. The polynomial function of Example 4 has three real zeros, and its factored form contains three linear factors. The polynomial function of Example 5 has two distinct real zeros, and its factored form contains two distinct linear factors and one irreducible quadratic factor.

**Theorem**

Every polynomial function (with real coefficients) can be uniquely factored into a product of linear factors and/or irreducible quadratic factors.

We shall prove this result in Section 3.7, and, in fact, we shall draw several additional conclusions about the zeros of a polynomial function. One conclusion is worth noting now. If a polynomial (with real coefficients) is of odd degree, then it must contain at least one linear factor. (Do you see why?) This means that it must have at least one real zero.

**Corollary**

A polynomial function (with real coefficients) of odd degree has at least one real zero.

**5 Use the Theorem for Bounds on Zeros**

One challenge in using a graphing utility is to set the viewing window so that a complete graph is obtained. The next theorem is a tool that can be used to find bounds on the zeros. This will assure that the function does not have any zeros outside these bounds. Then using these bounds to set \( X_{\text{min}} \) and \( X_{\text{max}} \) assures that all the \( x \)-intercepts appear in the viewing window.

A number \( M \) is a bound on the zeros of a polynomial if every zero \( r \) lies between \(-M\) and \( M \), inclusive. That is, \( M \) is a bound to the zeros of a polynomial \( f \) if

\[
-M \leq \text{any zero of } f \leq M
\]

**Theorem**

**Bounds on Zeros**

Let \( f \) denote a polynomial function whose leading coefficient is 1.

\[
f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0
\]

A bound \( M \) on the zeros of \( f \) is the smaller of the two numbers

\[
M = \max\{1, |a_0| + |a_1| + \cdots + |a_{n-1}|\}, \quad 1 + M = \max\{|a_0|, |a_1|, \ldots, |a_{n-1}|\} \quad (4)
\]

where \( \max \{ \} \) means "choose the largest entry in \{ \} ."

A n example will help to make the theorem clear.
Using the Theorem for Finding Bounds on Zeros

Find a bound to the zeros of each polynomial.

(a) \( f(x) = x^5 + 3x^3 - 9x^2 + 5 \)  \hspace{1cm} (b) \( g(x) = 4x^5 - 2x^3 + 2x^2 + 1 \)

**Solution**

(a) The leading coefficient of \( f \) is 1.

\[
f(x) = x^5 + 3x^3 - 9x^2 + 5 \quad \text{with} \quad a_4 = 0, a_3 = 3, a_2 = -9, a_1 = 0, a_0 = 5
\]

We evaluate the expressions in formula (4).

\[
\text{Max}\{1, |a_0| + |a_1| + \cdots + |a_{n-1}|\} = \text{Max}\{1, 5 + |0| + |9| + |3| + |0|\} = \text{Max}\{1, 17\} = 17
\]

\[
1 + \text{Max}\{|a_0|, |a_1|, \ldots, |a_{n-1}|\} = 1 + \text{Max}\{|5, 0, 9, 3, 0\} = 1 + 9 = 10
\]

The smaller of the two numbers, 10, is the bound. Every zero of \( f \) lies between \(-10 \) and \( 10 \).

(b) First we write \( g \) so that its leading coefficient is 1.

\[
g(x) = 4x^5 - 2x^3 + 2x^2 + 1 = 4\left(x^5 - \frac{1}{2}x^3 + \frac{1}{2}x^2 + \frac{1}{4}\right)
\]

Next we evaluate the two expressions in formula (4) with \( a_4 = 0, a_3 = -\frac{1}{2}, \)

\[
a_2 = \frac{1}{2}, a_1 = 0, \text{ and } a_0 = \frac{1}{4}.
\]

\[
\text{Max}\{1, |a_0| + |a_1| + \cdots + |a_{n-1}|\} = \text{Max}\{1, \left|\frac{1}{4}\right| + |0| + \left|\frac{1}{2}\right| + \left|-\frac{1}{2}\right| + |0|\} = \text{Max}\{1, \frac{5}{4}\} = \frac{5}{4}
\]

\[
1 + \text{Max}\{|a_0|, |a_1|, \ldots, |a_{n-1}|\} = 1 + \text{Max}\{\left|\frac{1}{4}\right|, |0|, \left|\frac{1}{2}\right|, \left|-\frac{1}{2}\right|, |0|\} = 1 + \frac{1}{2} = \frac{3}{2}
\]

The smaller of the two numbers, \( \frac{3}{2} \), is the bound. Every zero of \( g \) lies between \( -\frac{5}{4} \) and \( \frac{5}{4} \).

**EXAMPLE 8**

Obtaining Graphs Using Bounds on Zeros

Obtain a graph for each polynomial.

(a) \( f(x) = x^5 + 3x^3 - 9x^2 + 5 \)  \hspace{1cm} (b) \( g(x) = 4x^5 - 2x^3 + 2x^2 + 1 \)
(a) Based on Example 7(a), every zero lies between $-10$ and $10$. Using $X_{\text{min}} = -10$ and $X_{\text{max}} = 10$, we graph $Y_1 = f(x) = x^5 + 3x^3 - 9x^2 + 5$. Figure 76(a) shows the graph obtained using ZOOM-FIT. Figure 76(b) shows the graph after adjusting the viewing window to improve the graph.

(b) Based on Example 7(b), every zero lies between $-\frac{5}{4}$ and $\frac{5}{4}$. Using $X_{\text{min}} = -\frac{5}{4}$ and $X_{\text{max}} = \frac{5}{4}$, we graph $Y_1 = g(x) = 4x^5 - 2x^3 + 2x^2 + 1$. Figure 77 shows the graph after using ZOOM-FIT. Here no adjustment of the viewing window is needed.

Now work Problem 33.

The next example shows how to proceed when some of the coefficients of the polynomial are not integers.

**Example 9**

Finding the Zeros of a Polynomial

Find all the real zeros of the polynomial function

$$f(x) = x^5 - 1.8x^4 - 17.79x^3 + 31.672x^2 + 37.95x - 8.7121$$

**Step 1:** There are at most five real zeros.

**Step 2:** Since there are noninteger coefficients, the Rational Zeros Theorem does not apply.

**Step 3:** We determine the bounds of $f$. The leading coefficient of $f$ is 1 with $a_4 = -1.8$, $a_3 = -17.79$, $a_2 = 31.672$, $a_1 = 37.95$, and $a_0 = -8.7121$. We evaluate the expressions using formula (4).

$$\max\{1, \ | - 8.7121| + |37.95| + |31.672| + |-17.79| + |-1.8| \} = \max\{1, 97.9241\} = 97.9241$$

$$1 + \max\{ | - 8.7121|, |37.95|, |31.672|, |-17.79|, |-1.8| \} = 1 + 37.95 = 38.95$$

The smaller of the two numbers, 38.95, is the bound. Every real zero of $f$ lies between $-38.95$ and $38.95$. Figure 78(a) shows the graph of $f$ with $X_{\text{min}} = -38.95$ and $X_{\text{max}} = 38.95$. Figure 78(b) shows a graph of $f$ after adjusting the viewing window to improve the graph.

**Step 4:** From Figure 78(b), we see that $f$ appears to have four $x$-intercepts: one near $-4$, one near $-1$, one between 0 and 1, and one near 3. The $x$-intercept near 3 might be a zero of even multiplicity since the graph seems to touch the $x$-axis at that point.
We use the Factor Theorem to determine if $-4$ and $-1$ are zeros. Since 
$f(-4) = f(-1) = 0$, we know that $-4$ and $-1$ are zeros. Using ZERO (or 
ROOT), we find that the remaining zeros are 0.20 and 3.30, rounded to two 
decimal places.

There are no real zeros on the graph that have not already been identi-
fied. So, either 3.30 is a zero of multiplicity 2 or there are two distinct zeros, 
each of which is 3.30, rounded to two decimal places. (Example 10 will 
explain how to determine which is true.)

6 Use the Intermediate Value Theorem

The Intermediate Value Theorem requires that the function be continuous. A lthough calculus is needed to explain the meaning precisely, the idea of a continuous function 
is easy to understand. Very basically, a function $f$ is continuous when its graph can be 
drawn without lifting pencil from paper, that is, when the graph contains no holes or 
jumps or gaps. For example, every polynomial function is continuous.

Let $f$ denote a continuous function. If $a < b$ and if $f(a)$ and $f(b)$ are of 
opposite sign, then $f$ has at least one zero between $a$ and $b$.

A lthough the proof of this result requires advanced methods in calculus, it is 
easy to “see” why the result is true. Look at Figure 79.

Figure 79
If $f(a) < 0$ and $f(b) > 0$ and if $f$ is 
continuous, there is at least one zero between $a$ and $b$.

The Intermediate Value Theorem together with the TABLE feature of a 
graphing utility provides a basis for finding zeros.

EXAMPLE 10 Using the Intermediate Value Theorem and a Graphing Utility to 
Locate Zeros

Continue working with Example 9 to determine whether there is a repeated zero or 
two distinct zeros near 3.30.

We use the TABLE feature of a graphing utility. See Table 21. Since 
f(3.29) = 0.00956 > 0 and $f(3.30) = -0.0001 < 0$, by the Intermediate Value 
Theorem there is a zero between 3.29 and 3.30. Similarly, since 
f(3.30) = -0.0001 < 0 and $f(3.31) = 0.0097 > 0$, there is another zero between 
3.30 and 3.31. Now we know that the five zeros of $f$ are distinct.

NOW WORK PROBLEM 75.
CHAPTER 3 Polynomial and Rational Functions

Formulas for the solution of third- and fourth-degree polynomial equations exist, and, while not very practical, they do have an interesting history.

In the 1500s in Italy, mathematical contests were a popular pastime, and persons possessing methods for solving problems kept them secret. (Solutions that were published were already common knowledge.) Niccolo of Brescia (1500–1557), commonly referred to as Tartaglia ("the stammerer"), had the secret for solving cubic (third-degree) equations, which gave him a decided advantage in the contests. See the Historical Problems.

Girolamo Cardano (1501–1576) found out that Tartaglia had the secret, and, being interested in cubics, he requested it from Tartaglia. The reluctant Tartaglia hesitated for some time, but finally, swearing Cardano to secrecy with midnight oaths by candlelight, told him the secret. Cardano then published the solution in his book Ars Magna (1545), giving Tartaglia the credit but rather compromising the secrecy. Tartaglia exploded into bitter recriminations, and each wrote pamphlets that reflected on the other’s mathematics, moral character, and ancestry. See the Historical Problems.

The quartic (fourth-degree) equation was solved by Cardano’s student Lodovico Ferrari, and this solution also was included, with credit and this time with permission, in the Ars Magna.

Attempts were made to solve the fifth-degree equation in similar ways, all of which failed. In the early 1800s, P. Ruffini, Niels Abel, and Evariste Galois all found ways to show that it is not possible to solve fifth-degree equations by formula, but the proofs required the introduction of new methods. Galois’s methods eventually developed into a large part of modern algebra.

Problems 1–8 develop the Tartaglia–Cardano solution of the cubic equation and show why it is not altogether practical.

1. Show that the general cubic equation $y^3 + by^2 + cy + d = 0$ can be transformed into an equation of the form $x^3 + px + q = 0$ by using the substitution $y = x - \frac{b}{3}$.

2. In the equation $x^3 + px + q = 0$, replace $x$ by $H + K$. Let $3HK = -p$, and show that $H^3 + K^3 = -q$.

3. Based on Problem 2, we have the two equations

\[ 3HK = -p \quad \text{and} \quad H^3 + K^3 = -q. \]

Solve for $K$ in $3HK = -p$ and substitute into $H^3 + K^3 = -q$. Then show that

\[ H = \sqrt[3]{\frac{-q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} \]

[HINT: Look for an equation that is quadratic in form.]

4. Use the solution for $H$ from Problem 3 and the equation $H^3 + K^3 = -q$ to show that

\[ K = \sqrt[3]{\frac{-q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} \]

5. Use the results from Problems 2–4 to show that the solution of $x^3 + px + q = 0$ is

\[ x = \sqrt[3]{\frac{-q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} + \sqrt[3]{\frac{-q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} \]

6. Use the result of Problem 5 to solve the equation $x^3 - 6x - 9 = 0$.

7. Use a calculator and the result of Problem 5 to solve the equation $x^3 + 3x - 14 = 0$.

8. Use the methods of this chapter to solve the equation $x^3 - 6x - 9 = 0$.

Historical Problems

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Tartaglia (1500–1557)