3.1 Quadratic Functions and Models

PREPARING FOR THIS SECTION

Before getting started, review the following:

• Intercepts (Section 1.2, pp. 15–17)
• Quadratic Equations (Appendix, Section A.5, pp. 988–995)
• Completing the Square (Appendix, Section A.5, pp. 991–992)
• Graphing Techniques: Transformations (Section 2.6, pp. 118–126)

Now work the ‘Are You Prepared?’ problems on page 163.

OBJECTIVES

1. Graph a Quadratic Function Using Transformations
2. Identify the Vertex and Axis of Symmetry of a Quadratic Function
3. Graph a Quadratic Function Using Its Vertex, Axis, and Intercepts
4. Use the Maximum or Minimum Value of a Quadratic Function to Solve Applied Problems
5. Use a Graphing Utility to Find the Quadratic Function of Best Fit to Data

A quadratic function is a function that is defined by a second-degree polynomial in one variable.

A quadratic function is a function of the form

\[ f(x) = ax^2 + bx + c \]  

where \( a, b, \) and \( c \) are real numbers and \( a \neq 0 \). The domain of a quadratic function is the set of all real numbers.

Many applications require a knowledge of quadratic functions. For example, suppose that Texas Instruments collects the data shown in Table 1, which relate the number of calculators sold at the price \( p \) (in dollars) per calculator. Since the price of a product determines the quantity that will be purchased, we treat price as the independent variable. The relationship between the number \( x \) of calculators sold and the price \( p \) per calculator may be approximated by the linear equation

\[ x = 21,000 - 150p \]

<table>
<thead>
<tr>
<th>Price per Calculator, ( p ) (Dollars)</th>
<th>Number of Calculators, ( x )</th>
</tr>
</thead>
<tbody>
<tr>
<td>60</td>
<td>11,100</td>
</tr>
<tr>
<td>65</td>
<td>10,115</td>
</tr>
<tr>
<td>70</td>
<td>9,652</td>
</tr>
<tr>
<td>75</td>
<td>8,731</td>
</tr>
<tr>
<td>80</td>
<td>8,087</td>
</tr>
<tr>
<td>85</td>
<td>7,205</td>
</tr>
<tr>
<td>90</td>
<td>6,439</td>
</tr>
</tbody>
</table>

Then the revenue \( R \) derived from selling \( x \) calculators at the price \( p \) per calculator is equal to the unit selling price \( p \) of the product times the number \( x \) of units actually sold. That is,
SECTION 3.1 Quadratic Functions and Models 151

So, the revenue \( R \) is a quadratic function of the price \( p \). Figure 1 illustrates the graph of this revenue function, whose domain is \( 0 \leq p \leq 140 \), since both \( x \) and \( p \) must be non-negative. Later in this section we shall determine the price \( p \) that maximizes revenue.

A second situation in which a quadratic function appears involves the motion of a projectile. Based on Newton’s second law of motion (force equals mass times acceleration, \( F = ma \)), it can be shown that, ignoring air resistance, the path of a projectile propelled upward at an inclination to the horizontal is the graph of a quadratic function. See Figure 2 for an illustration. Later in this section we shall analyze the path of a projectile.

Figure 1
Path of a cannonball

Graph a Quadratic Function Using Transformations

We know how to graph the quadratic function \( f(x) = x^2 \). Figure 3 shows the graph of three functions of the form \( f(x) = ax^2, a > 0 \), for \( a = 1, a = \frac{1}{2}, \) and \( a = 3 \). Notice that the larger the value of \( a \), the “narrower” the graph is, and the smaller the value of \( a \), the “wider” the graph is.

Figure 3
Graphs of a quadratic function, \( f(x) = ax^2 + bx + c, a \neq 0 \)

Figure 4 shows the graphs of \( f(x) = ax^2 \) for \( a < 0 \). Notice that these graphs are reflections about the x-axis of the graphs in Figure 3. Based on the results of these two figures, we can draw some general conclusions about the graph of \( f(x) = ax^2 \).

First, as \( |a| \) increases, the graph becomes “taller” (a vertical stretch), and as \( |a| \) gets closer to zero, the graph gets “shorter” (a vertical compression). Second, if \( a \) is positive, then the graph opens “up,” and if \( a \) is negative, then the graph opens “down.”

The graphs in Figures 3 and 4 are typical of the graphs of all quadratic functions, which we call parabolas. Refer to Figure 5, where two parabolas are pictured. The one on the left opens up and has a lowest point; the one on the right opens down and has a highest point. The lowest or highest point of a parabola is called the vertex.

*We shall study parabolas using a geometric definition later in this book.
The vertical line passing through the vertex in each parabola in Figure 5 is called the **axis of symmetry** (usually abbreviated to **axis**) of the parabola. Because the parabola is symmetric about its axis, the axis of symmetry of a parabola can be used to find additional points on the parabola when graphing by hand.

The parabolas shown in Figure 5 are the graphs of a quadratic function $f(x) = ax^2 + bx + c, a \neq 0$. Notice that the coordinate axes are not included in the figure. Depending on the values of $a$, $b$, and $c$, the axes could be placed anywhere. The important fact is that the shape of the graph of a quadratic function will look like one of the parabolas in Figure 5.

In the following example, we use techniques from Section 2.6 to graph a quadratic function $f(x) = ax^2 + bx + c, a \neq 0$. In so doing, we shall complete the square and write the function $f$ in the form $f(x) = a(x - h)^2 + k$.

**EXAMPLE 1**  
**Graphing a Quadratic Function Using Transformations**

Graph the function $f(x) = 2x^2 + 8x + 5$. Find the vertex and axis of symmetry.

**Solution**  
We begin by completing the square on the right side.

$$f(x) = 2x^2 + 8x + 5$$

$$= 2(x^2 + 4x) + 5$$  
**Factor out the 2 from $2x^2 + 8x$.**

$$= 2(x^2 + 4x + 4) + 5 - 8$$  
**Complete the square of $2(x^2 + 4x)$, Notice that the factor of 2 requires that 8 be added and subtracted.**

$$= 2(x + 2)^2 - 3$$  

(2)

The graph of $f$ can be obtained in three stages, as shown in Figure 6. Now compare this graph to the graph in Figure 5(a). The graph of $f(x) = 2x^2 + 8x + 5$ is a parabola that opens up and has its vertex (lowest point) at $(-2, -3)$. Its axis of symmetry is the line $x = -2$.

**Figure 6**

(a) $y = x^2$  
Multiply by 2; Vertical stretch  
(b) $y = 2x^2$  
Replace $x$ by $x + 2$; Shift left 2 units  
(c) $y = 2(x + 2)^2$  
Subtract 3; Shift down 3 units  
(d) $y = 2(x + 2)^2 - 3$

✔ **CHECK:** Use a graphing utility to graph $f(x) = 2x^2 + 8x + 5$ and use the MINIMUM command to locate its vertex.

**NOW WORK PROBLEM 27.**

The method used in Example 1 can be used to graph any quadratic function $f(x) = ax^2 + bx + c, a \neq 0$, as follows:
\[ f(x) = ax^2 + bx + c \]
\[ = a\left(x^2 + \frac{b}{a}x\right) + c \quad \text{Factor out } a \text{ from } ax^2 + bx. \]
\[ = a\left(x^2 + \frac{b}{a}x + \frac{b^2}{4a^2}\right) + c - a\left(\frac{b^2}{4a^2}\right) \quad \text{Complete the square by adding and subtracting } \left(\frac{b^2}{4a^2}\right). \text{ Look closely at this step!} \]
\[ = a\left(x + \frac{b}{2a}\right)^2 + c - \frac{b^2}{4a} \quad \text{Factor} \]
\[ = a\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a} \quad c - \frac{b^2}{4a} = c \cdot \frac{4a}{4a} - \frac{b^2}{4a} = \frac{4ac - b^2}{4a} \]

Based on these results, we conclude the following:

If \( h = -\frac{b}{2a} \) and \( k = \frac{4ac - b^2}{4a} \), then

\[ f(x) = ax^2 + bx + c = a(x - h)^2 + k \tag{3} \]

The graph of \( f(x) = a(x - h)^2 + k \) is the parabola \( y = ax^2 \) shifted horizontally \( h \) units (replace \( x \) by \( x - h \)) and vertically \( k \) units (add \( k \)). As a result, the vertex is at \( (h, k) \), and the graph opens up if \( a > 0 \) and down if \( a < 0 \). The axis of symmetry is the vertical line \( x = h \).

For example, compare equation (3) with equation (2) of Example 1.

\[ f(x) = 2(x + 2)^2 - 3 \]
\[ = 2(x - (-2))^2 - 3 \]
\[ = a(x - h)^2 + k \]

We conclude that \( a = 2 \), so the graph opens up. Also, we find that \( h = -2 \) and \( k = -3 \), so its vertex is at \( (-2, -3) \).

2 Identify the Vertex and Axis of Symmetry of a Quadratic Function

We do not need to complete the square to obtain the vertex. In almost every case, it is easier to obtain the vertex of a quadratic function \( f \) by remembering that its \( x \)-coordinate is \( h = -\frac{b}{2a} \). The \( y \)-coordinate can then be found by evaluating \( f \) at \( -\frac{b}{2a} \) to find \( k = f\left(-\frac{b}{2a}\right)\).

We summarize these remarks as follows:

**Properties of the Graph of a Quadratic Function**

\[ f(x) = ax^2 + bx + c, \quad a \neq 0 \]

Vertex = \( \left(-\frac{b}{2a}, f\left(-\frac{b}{2a}\right)\right) \) \quad A axis of symmetry: the line \( x = -\frac{b}{2a} \) \tag{4}

Parabola opens up if \( a > 0 \); the vertex is a minimum point.
Parabola opens down if \( a < 0 \); the vertex is a maximum point.
EXAMPLE 2

Locating the Vertex without Graphing

Without graphing, locate the vertex and axis of symmetry of the parabola defined by
\[ f(x) = -3x^2 + 6x + 1. \] Does it open up or down?

Solution

For this quadratic function, \( a = -3, \ b = 6, \) and \( c = 1. \) The x-coordinate of the vertex is
\[ h = -\frac{b}{2a} = -\frac{6}{-6} = 1 \]
The y-coordinate of the vertex is
\[ k = f\left(\frac{-b}{2a}\right) = f(1) = -3 + 6 + 1 = 4 \]
The vertex is located at the point \((1, 4)\). The axis of symmetry is the line \( x = 1. \) Because \( a = -3 < 0, \) the parabola opens down.

Graph a Quadratic Function Using Its Vertex, Axis, and Intercepts

The information we gathered in Example 2, together with the location of the intercepts, usually provides enough information to graph \( f(x) = ax^2 + bx + c, \ a \neq 0, \) by hand.

The y-intercept is the value of \( f \) at \( x = 0; \) that is, \( f(0) = c. \)

The x-intercepts, if there are any, are found by solving the quadratic equation
\[ ax^2 + bx + c = 0 \]
This equation has two, one, or no real solutions, depending on whether the discriminant \( b^2 - 4ac \) is positive, 0, or negative. The graph of \( f \) has x-intercepts, as follows:

The x-intercepts of a Quadratic Function

1. If the discriminant \( b^2 - 4ac > 0, \) the graph of \( f(x) = ax^2 + bx + c \) has two distinct x-intercepts and so will cross the x-axis in two places.
2. If the discriminant \( b^2 - 4ac = 0, \) the graph of \( f(x) = ax^2 + bx + c \) has one x-intercept and touches the x-axis at its vertex.
3. If the discriminant \( b^2 - 4ac < 0, \) the graph of \( f(x) = ax^2 + bx + c \) has no x-intercept and so will not cross or touch the x-axis.

Figure 7 illustrates these possibilities for parabolas that open up.
Graphing a Quadratic Function by Hand Using Its Vertex, Axis, and Intercepts

Use the information from Example 2 and the locations of the intercepts to graph \( f(x) = -3x^2 + 6x + 1 \). Determine the domain and the range of \( f \). Determine where \( f \) is increasing and where it is decreasing.

**Solution**

In Example 2, we found the vertex to be at \((1, 4)\) and the axis of symmetry to be \( x = 1 \). The \( y \)-intercept is found by letting \( x = 0 \). The \( y \)-intercept is \( f(0) = 1 \). The \( x \)-intercepts are found by solving the equation \( f(x) = 0 \). This results in the equation

\[ -3x^2 + 6x + 1 = 0 \quad a = -3, b = 6, c = 1 \]

The discriminant \( b^2 - 4ac = (6)^2 - 4(-3)(1) = 36 + 12 = 48 > 0 \), so the equation has two real solutions and the graph has two \( x \)-intercepts. Using the quadratic formula, we find that

\[ x = \frac{-b + \sqrt{b^2 - 4ac}}{2a} = \frac{-6 + \sqrt{48}}{-6} = \frac{-6 + 4\sqrt{3}}{-6} \approx -0.15 \]

and

\[ x = \frac{-b - \sqrt{b^2 - 4ac}}{2a} = \frac{-6 - \sqrt{48}}{-6} = \frac{-6 - 4\sqrt{3}}{-6} \approx 2.15 \]

The \( x \)-intercepts are approximately \(-0.15\) and \(2.15\).

The graph is illustrated in Figure 8. Notice how we used the \( y \)-intercept and the axis of symmetry, to obtain the additional point on the graph.

The domain of \( f \) is the set of all real numbers. Based on the graph, the range of \( f \) is the interval \((−∞, 4]\). The function \( f \) is increasing on the interval \((−∞, 1)\) and decreasing on the interval \((1, ∞)\).

✔ **CHECK:** Graph \( f(x) = -3x^2 + 6x + 1 \) using a graphing utility. Use **ZERO** (or **ROOT**) to locate the two \( x \)-intercepts. Use **MAXIMUM** to locate the vertex.

NOW WORK PROBLEM 35.

If the graph of a quadratic function has only one \( x \)-intercept or no \( x \)-intercepts, it is usually necessary to plot an additional point to obtain the graph by hand.

Graphing a Quadratic Function by Hand Using Its Vertex, Axis, and Intercepts

Graph \( f(x) = x^2 - 6x + 9 \) by determining whether the graph opens up or down and by finding its vertex, axis of symmetry, \( y \)-intercept, and \( x \)-intercepts, if any. Determine the domain and the range of \( f \). Determine where \( f \) is increasing and where it is decreasing.

**Solution**

For \( f(x) = x^2 - 6x + 9 \), we have \( a = 1, b = -6, \) and \( c = 9 \). Since \( a = 1 > 0 \), the parabola opens up. The \( x \)-coordinate of the vertex is

\[ h = -\frac{b}{2a} = -\frac{-6}{2(1)} = 3 \]
The y-coordinate of the vertex is
\[ k = f(3) = (3)^2 - 6(3) + 9 = 0 \]
So, the vertex is at (3, 0). The axis of symmetry is the line \( x = 3 \). The y-intercept is \( f(0) = 9 \). Since the vertex (3, 0) lies on the x-axis, the graph touches the x-axis at the x-intercept. By using the axis of symmetry and the y-intercept at (0, 9), we can locate the additional point (6, 9) on the graph. See Figure 9.

The domain of \( f \) is the set of all real numbers. Based on the graph, the range of \( f \) is the interval \([0, \infty)\). The function \( f \) is decreasing on the interval \((-\infty, 3)\) and increasing on the interval \((3, \infty)\).

Graph the function in Example 4 by completing the square and using transformations. Which method do you prefer?

**NOW WORK PROBLEM 43.**

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**Graphing a Quadratic Function by Hand Using Its Vertex, Axis, and Intercepts**

Graph \( f(x) = 2x^2 + x + 1 \) by determining whether the graph opens up or down and by finding its vertex, axis of symmetry, y-intercept, and x-intercepts, if any. Determine the domain and the range of \( f \). Determine where \( f \) is increasing and where it is decreasing.

For \( f(x) = 2x^2 + x + 1 \), we have \( a = 2 \), \( b = 1 \), and \( c = 1 \). Since \( a = 2 > 0 \), the parabola opens up. The x-coordinate of the vertex is
\[ h = -\frac{b}{2a} = -\frac{1}{4} \]

The y-coordinate of the vertex is
\[ k = f\left(-\frac{1}{4}\right) = 2\left(\frac{1}{16}\right) + \left(-\frac{1}{4}\right) + 1 = \frac{7}{8} \]
So, the vertex is at \( \left(-\frac{1}{4}, \frac{7}{8}\right) \). The axis of symmetry is the line \( x = -\frac{1}{4} \). The y-intercept is \( f(0) = 1 \). The x-intercept(s), if any, obey the equation \( 2x^2 + x + 1 = 0 \). Since the discriminant \( b^2 - 4ac = (1)^2 - 4(2)(1) = -7 < 0 \), this equation has no real solutions, and therefore the graph has no x-intercepts. We use the point (0, 1) and the axis of symmetry \( x = -\frac{1}{4} \) to locate the additional point \( \left(-\frac{1}{2}, 1\right) \) on the graph. See Figure 10.

The domain of \( f \) is the set of all real numbers. Based on the graph, the range of \( f \) is the interval \( \left[\frac{7}{8}, \infty\right) \). The function \( f \) is decreasing on the interval \( (-\infty, -\frac{1}{4}) \) and increasing on the interval \( \left(-\frac{1}{4}, \infty\right) \).

**NOW WORK PROBLEM 47.**

Given the vertex \((h, k)\) and one additional point on the graph of a quadratic function \( f(x) = ax^2 + bx + c, a \neq 0 \), we can use

\[ f(x) = a(x - h)^2 + k \quad (5) \]

to obtain the quadratic function.
SECTION 3.1 Quadratic Functions and Models

Finding the Quadratic Function Given Its Vertex and One Other Point

Determine the quadratic function whose vertex is \((1, -5)\) and whose y-intercept is \(-3\). The graph of the parabola is shown in Figure 11.

**Solution**
The vertex is \((1, -5)\), so \(h = 1\) and \(k = -5\). Substitute these values into equation (5).

\[
f(x) = a(x - h)^2 + k \quad \text{Equation (5)}
\]

\[
f(x) = a(x - 1)^2 - 5
d = 1, k = -5
\]

To determine the value of \(a\), we use the fact that \((0, -3)\) (the y-intercept).

\[
f(x) = a(x - 1)^2 - 5
-3 = a(0 - 1)^2 - 5
-3 = a - 5
a = 2
\]

The quadratic function whose graph is shown in Figure 11 is

\[
f(x) = 2(x - 1)^2 - 5 = 2x^2 - 4x - 3
\]

**CHECK:** Figure 12 shows the graph of \(f(x) = 2x^2 - 4x - 3\) using a graphing utility.

**Summary**

**Steps for Graphing a Quadratic Function** \(f(x) = ax^2 + bx + c, a \neq 0\), by Hand.

**Option 1**

**Step 1:** Complete the square in \(x\) to write the quadratic function in the form \(f(x) = a(x - h)^2 + k\).

**Step 2:** Graph the function in stages using transformations.

**Option 2**

**Step 1:** Determine the vertex \((-\frac{b}{2a}, -f(-\frac{b}{2a}))\).

**Step 2:** Determine the axis of symmetry, \(x = -\frac{b}{2a}\).

**Step 3:** Determine the y-intercept, \(f(0)\).

**Step 4:** (a) If \(b^2 - 4ac > 0\), then the graph of the quadratic function has two \(x\)-intercepts, which are found by solving the equation \(ax^2 + bx + c = 0\).

- (b) If \(b^2 - 4ac = 0\), the vertex is the \(x\)-intercept.

- (c) If \(b^2 - 4ac < 0\), there are no \(x\)-intercepts.

**Step 5:** Determine an additional point by using the y-intercept and the axis of symmetry.

**Step 6:** Plot the points and draw the graph.
4 Use the Maximum or Minimum Value of a Quadratic Function to Solve Applied Problems

When a mathematical model leads to a quadratic function, the properties of this quadratic function can provide important information about the model. For example, for a quadratic revenue function, we can find the maximum revenue; for a quadratic cost function, we can find the minimum cost.

To see why, recall that the graph of a quadratic function 

\[ f(x) = ax^2 + bx + c, \quad a \neq 0 \]

is a parabola with vertex at \( \left(-\frac{b}{2a}, f\left(-\frac{b}{2a}\right)\right) \). This vertex is the highest point on the graph if \( a < 0 \) and the lowest point on the graph if \( a > 0 \). If the vertex is the highest point \((a < 0)\), then \( f\left(-\frac{b}{2a}\right) \) is the maximum value of \( f \). If the vertex is the lowest point \((a > 0)\), then \( f\left(-\frac{b}{2a}\right) \) is the minimum value of \( f \).

This property of the graph of a quadratic function enables us to answer questions involving optimization (finding maximum or minimum values) in models involving quadratic functions.

**Example 7**

Finding the Maximum or Minimum Value of a Quadratic Function

Determine whether the quadratic function

\[ f(x) = x^2 - 4x - 5 \]

has a maximum or minimum value. Then find the maximum or minimum value.

**Solution**

We compare \( f(x) = x^2 - 4x - 5 \) to \( f(x) = ax^2 + bx + c \). We conclude that \( a = 1, b = -4, \) and \( c = -5 \). Since \( a > 0 \), the graph of \( f \) opens up, so the vertex is a minimum point. The minimum value occurs at

\[ x = -\frac{b}{2a} = -\frac{-4}{2(1)} = \frac{4}{2} = 2 \]

\[ a = 1, b = -4 \]

The minimum value is

\[ f\left(-\frac{b}{2a}\right) = f(2) = 2^2 - 4(2) - 5 = 4 - 8 - 5 = -9 \]

**Check:** We can support the algebraic solution using the TABLE feature on a graphing utility. Create Table 2 by letting \( Y_1 = x^2 - 4x - 5 \). From the table we see that the smallest value of \( y \) occurs when \( x = 2 \), leading us to investigate the number 2 further. The symmetry about \( x = 2 \) confirms that 2 is the \( x \)-coordinate of the vertex of the parabola. We conclude that the minimum value is \(-9\) and occurs at \( x = 2 \).

**Now work problem 61.**
EXAMPLE 8 Maximizing Revenue

The marketing department at Texas Instruments has found that, when certain calculators are sold at a price of \( p \) dollars per unit, the revenue \( R \) (in dollars) as a function of the price \( p \) is

\[
R(p) = -150p^2 + 21,000p
\]

What unit price should be established to maximize revenue? If this price is charged, what is the maximum revenue?

**Solution**

The revenue \( R \) is

\[
R(p) = -150p^2 + 21,000p
\]

The function \( R \) is a quadratic function with \( a = -150, b = 21,000, \) and \( c = 0. \) Because \( a < 0, \) the vertex is the highest point on the parabola. The revenue \( R \) is therefore a maximum when the price \( p \) is

\[
p = -\frac{b}{2a} = -\frac{21,000}{2(-150)} = -\frac{21,000}{-300} = 70.00
\]

The maximum revenue \( R \) is

\[
R(70) = -150(70)^2 + 21,000(70) = 735,000
\]

See Figure 13 for an illustration.

**Figure 13**

![Graph showing the revenue function and its maximum value.]

NOW WORK PROBLEM 71.

EXAMPLE 9 Maximizing the Area Enclosed by a Fence

A farmer has 2000 yards of fence to enclose a rectangular field. What are the dimensions of the rectangle that encloses the most area?

**Solution**

Figure 14 illustrates the situation. The available fence represents the perimeter of the rectangle. If \( x \) is the length and \( w \) is the width, then

\[
2x + 2w = 2000 \quad (6)
\]

The area \( A \) of the rectangle is

\[
A = xw
\]
To express $A$ in terms of a single variable, we solve equation (6) for $w$ and substitute the result in $A = xw$. Then $A$ involves only the variable $x$. [You could also solve equation (6) for $x$ and express $A$ in terms of $w$ alone. Try it!]

\[
2x + 2w = 2000 \quad \text{Equation (6)} \\
2w = 2000 - 2x \quad \text{Solve for } w. \\
w = \frac{2000 - 2x}{2} = 1000 - x
\]

Then the area $A$ is

\[
A = xw = x(1000 - x) = -x^2 + 1000x
\]

Now, $A$ is a quadratic function of $x$.

\[
A(x) = -x^2 + 1000x \quad a = -1, b = 1000, c = 0
\]

Figure 15 shows a graph of $A(x) = -x^2 + 1000x$ using a graphing utility. Since $a < 0$, the vertex is a maximum point on the graph of $A$. The maximum value occurs at

\[
x = -\frac{b}{2a} = -\frac{1000}{2(-1)} = 500
\]

The maximum value of $A$ is

\[
A\left(-\frac{b}{2a}\right) = A(500) = -500^2 + 1000(500) = -250,000 + 500,000 = 250,000
\]

The largest rectangle that can be enclosed by 2000 yards of fence has an area of 250,000 square yards. Its dimensions are 500 yards by 500 yards.

\[\text{NOW WORK PROBLEM 77.}\]

**EXAMPLE 10**

**Analyzing the Motion of a Projectile**

A projectile is fired from a cliff 500 feet above the water at an inclination of 45° to the horizontal, with a muzzle velocity of 400 feet per second. In physics, it is established that the height $h$ of the projectile above the water is given by

\[
h(x) = \frac{-32x^2}{(400)^2} + x + 500
\]

where $x$ is the horizontal distance of the projectile from the base of the cliff. See Figure 16.

(a) Find the maximum height of the projectile.

(b) How far from the base of the cliff will the projectile strike the water?
Solution

(a) The height of the projectile is given by a quadratic function.

\[ h(x) = \frac{-32x^2}{(400)^2} + x + 500 = \frac{-1}{5000}x^2 + x + 500 \]

We are looking for the maximum value of \( h \). Since \( a < 0 \), the maximum value is obtained at the vertex. We compute

\[ x = \frac{-b}{2a} = -\frac{1}{2 \left(-\frac{1}{5000}\right)} = \frac{5000}{2} = 2500 \]

The maximum height of the projectile is

\[ h(2500) = \frac{-1}{5000}(2500)^2 + 2500 + 500 = -1250 + 2500 + 500 = 1750 \text{ ft} \]

(b) The projectile will strike the water when the height is zero. To find the distance \( x \) traveled, we need to solve the equation

\[ h(x) = \frac{-1}{5000}x^2 + x + 500 = 0 \]

We find the discriminant first.

\[ b^2 - 4ac = 1^2 - 4\left(-\frac{1}{5000}\right)(500) = 1.4 \]

Then,

\[ x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-1 \pm \sqrt{1.4}}{2 \left(-\frac{1}{5000}\right)} \approx \begin{cases} -458 \\ 5458 \end{cases} \]

We discard the negative solution and find that the projectile will strike the water at a distance of about 5458 feet from the base of the cliff.

NOW WORK PROBLEM 81.

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**EXAMPLE 11**

**The Golden Gate Bridge**

The Golden Gate Bridge, a suspension bridge, spans the entrance to San Francisco Bay. Its 746-foot-tall towers are 4200 feet apart. The bridge is suspended from two huge cables more than 3 feet in diameter; the 90-foot-wide roadway is 220 feet above the water. The cables are parabolic in shape\(^1\) and touch the road surface at the center of the bridge. Find the height of the cable at a distance of 1000 feet from the center.

**Solution**

See Figure 17. We begin by choosing the placement of the coordinate axes so that the \( x \)-axis coincides with the road surface and the origin coincides with the center of the bridge. As a result, the twin towers will be vertical (height 746 – 220 = 526 feet above the road) and located 2100 feet from the center. Also, the cable, which has the shape of a catenary, but when a horizontal roadway is suspended from the cable, the cable takes the shape of a parabola.

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\(^1\) A cable suspended from two towers is in the shape of a catenary, but when a horizontal roadway is suspended from the cable, the cable takes the shape of a parabola.
shape of a parabola, will extend from the towers, open up, and have its vertex at (0, 0). The choice of placement of the axes enables us to identify the equation of the parabola as \( y = ax^2, a > 0 \). We can also see that the points (−2100, 526) and (2100, 526) are on the graph.

Based on these facts, we can find the value of \( a \) in \( y = ax^2 \).

\[
\begin{align*}
526 &= a(2100)^2 \\
a &= \frac{526}{(2100)^2}
\end{align*}
\]

The equation of the parabola is therefore

\[
y = \frac{526}{(2100)^2}x^2
\]

The height of the cable when \( x = 1000 \) is

\[
y = \frac{526}{(2100)^2}(1000)^2 \approx 119.3 \text{ feet}
\]

The cable is 119.3 feet high at a distance of 1000 feet from the center of the bridge.

Now work Problem 83.

5 Use a Graphing Utility to Find the Quadratic Function of Best Fit to Data

In Section 2.4, we found the line of best fit for data that appeared to be linearly related. It was noted that data may also follow a nonlinear relation. Figures 18(a) and (b) show scatter diagrams of data that follow a quadratic relation.

**Example 12**

Fitting a Quadratic Function to Data

A farmer collected the data given in Table 3, which shows crop yields \( Y \) for various amounts of fertilizer used, \( x \).

(a) With a graphing utility, draw a scatter diagram of the data. Comment on the type of relation that may exist between the two variables.

(b) Use a graphing utility to find the quadratic function of best fit to these data.

(c) Use the function found in part (b) to determine the optimal amount of fertilizer to apply.

**You Try It**
(d) Use the function found in part (b) to predict crop yield when the optimal amount of fertilizer is applied.

(e) Draw the quadratic function of best fit on the scatter diagram.

**Solution**

(a) Figure 19 shows the scatter diagram, from which it appears that the data follow a quadratic relation, with $a < 0$.

(b) Upon executing the QUADRATIC REGression program, we obtain the results shown in Figure 20. The output that the utility provides shows us the equation

$$y = ax^2 + bx + c.$$  

The quadratic function of best fit is

$$Y(x) = -0.0171x^2 + 1.0765x + 3.8939$$

where $x$ represents the amount of fertilizer used and $Y$ represents crop yield.

(c) Based on the quadratic function of best fit, the optimal amount of fertilizer to apply is

$$x = -\frac{b}{2a} = -\frac{1.0765}{2(-0.0171)} \approx 31.5$$

pounds of fertilizer per 100 square feet.

(d) We evaluate the function $Y(x)$ for $x = 31.5$.

$$Y(31.5) = -0.0171(31.5)^2 + 1.0765(31.5) + 3.8939 \approx 20.8$$

bushels

If we apply 31.5 pounds of fertilizer per 100 square feet, the crop yield will be 20.8 bushels according to the quadratic function of best fit.

(e) Figure 21 shows the graph of the quadratic function found in part (b) drawn on the scatter diagram.

Look again at Figure 20. Notice that the output given by the graphing calculator does not include $r$, the correlation coefficient. Recall that the correlation coefficient is a measure of the strength of a linear relation that exists between two variables. The graphing calculator does not provide an indication of how well the function fits the data in terms of $r$ since a quadratic function cannot be expressed as a linear function.

**NOW WORK PROBLEM 101.**