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Fixed points and stability in differential equations with variable delays

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Abstract

In this paper we consider a linear scalar differential equation with variable delays and give conditions to ensure that the zero solution is asymptotically stable by means of fixed point theory. These conditions do not require the boundedness of delays, nor do they ask for a fixed sign on the coefficient functions. An asymptotic stability theorem with a necessary and sufficient condition is proved.

Keywords: Fixed points; Stability; Delay equations; Variable delays

1. Introduction

We consider the scalar delay equation

$$x'(t) = -b(t)x(t - \tau(t)) \tag{1.1}$$

and its generalization

$$x'(t) = -\sum_{j=1}^{N} b_j(t)x(t - \tau_j(t)), \tag{1.2}$$

where $b, b_j \in C(R^+, R)$ and $\tau, \tau_j \in C(R^+, R^+)$ with $t - \tau(t) \to \infty$ and $t - \tau_j(t) \to \infty$ as $t \to \infty$. Note that (1.2) becomes (1.1) for $N = 1$.

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Delay differential equations of the type considered here arise in a variety of applications including control systems, electrodynamics, mixing liquids, neutron transportation, population models. It is the stability and asymptotic behavior of solutions of these model equations that is especially important to many investigators. For the historical background and discussions of applications to dynamical models, we refer the reader to, for example, the work of Burton and Haddock [4], Yorke [10], and the references contained therein.

This part of investigation is motivated by a number of difficulties encountered in the study of stability by means of Liapunov's direct method. We notice that these difficulties vanish when applying fixed point theory. While Liapunov's direct method usually requires pointwise conditions, the stability result we offer asks conditions of an averaging nature.

In the case of bounded delays, the stability and asymptotic behavior of solutions of (1.1), as well as more general cases, have been studied by many authors. It is well known that if there are positive numbers β and q such that

$$0 < b(t) \leqslant \beta, \quad \tau(t) \leqslant q \quad \text{and} \quad \beta q \leqslant \tfrac{3}{2}, \tag{1.3}$$

then the zero solution of (1.1) is stable (see [10]). Furthermore, the upper bound $\frac{3}{2}$ is sharp in the sense that if $\beta q > \frac{3}{2}$, there are equations with unbounded solutions. Yoneyama [9] generalized (1.3) to

$$\inf_{t \geqslant 0} \int_t^{t+q} b(s)\,\mathrm{d}s > 0, \quad \tau(t) \leqslant q, \quad \sup_{t \geqslant 0} \int_t^{t+q} b(s)\,\mathrm{d}s < \frac{3}{2} \tag{1.4}$$

and showed that the zero solution of (1.1) is uniformly asymptotically stable under (1.4). Krisztin [7] gave a generalization of Yorke's theorem with conditions flexible for more delays. A simplified version of his theorem may be stated as follows.

Theorem A (*Krisztin [7]*). *Suppose that* $b_i \in C(R^+, R^+)$, $\tau_i \in C(R^+, [0, q_i])$, *and*

$$0 < \alpha \leqslant b_i(t) \leqslant \beta_i, \quad \tau_i(t) \leqslant q_i, \quad \sum_{i=1}^{N} \beta_i q_i < 1. \tag{1.5}$$

Then the zero solution of (1.2) *is uniformly asymptotically stable.*

It is noted in [7] that the last estimate (upper bound) in (1.5) is the best possible for $N \geqslant 2$. Applying Liapunov's direct method, Kolmanovskii et al. [6] proved the following result.

Theorem B (*Kolmanovskii [6]*). *Assume that* τ *is differentiable*, $\tau'(t) \leqslant r < 1$ *for all* $t \geqslant 0$, $\inf_{t \geqslant 0} b(t) = \beta > 0$, *and*

$$\sup_{t \geqslant 0} \int_t^{g(t)} b(s)\,\mathrm{d}s = \mu < 1, \tag{1.6}$$

where $g(t)$ *is the inverse function of* $t - \tau(t)$. *Then the zero solution of* (1.1) *is uniformly stable. Moreover, if* $b(t)$ *is bounded, then the zero solution of* (1.1) *is uniformly asymptotically stable.*

Observe that the condition $\tau'(t) \leqslant r < 1$ implies that $t - \tau(t) \to \infty$ as $t \to \infty$ and that (1.6) is equivalent to

$$\sup_{t \geqslant u^*} \int_{t-\tau(t)}^{t} b(s)\,\mathrm{d}s = \mu < 1,$$

where u^* is the unique solution of $u - \tau(u) = 0$. This also yields the fact that $\tau(t)$ is bounded since $\inf_{t \geqslant 0} b(t) = \beta > 0$. In a paper by Graef et al. [5], boundedness and stability are obtained without asking that $\tau'(t) \leqslant r < 1$, $\inf_{t \geqslant 0} b(t) = \beta > 0$, or that $b(t)$ be bounded above.

Theorem C (*Graef [5]*). *Suppose that* $b(t) \geqslant 0$ *for all* $t \geqslant 0$, $t - \tau(t) \to \infty$ *as* $t \to \infty$, *and*

$$\limsup_{t \to \infty} \int_{t-\tau(t)}^{t} b(s)\,\mathrm{d}s = \mu < 1.$$

Then the zero solution of (1.1) *is asymptotically stable iff* $\int_0^\infty b(s)\,\mathrm{d}t = \infty$.

Note that the sign condition $b(t) \geqslant 0$ is required by all theorems mentioned above. Burton [2] eliminated this condition for the case $\tau(t) = r$, a constant, by applying fixed point theory with an appropriate mapping function.

Theorem D (*Burton [2]*). *Suppose that* $\tau(t) = r$ *and there exists a constant* $\alpha < 1$ *such that*

$$\int_{t-r}^{t} |b(s+r)|\,\mathrm{d}s + \int_0^t |b(s+r)|\,\mathrm{e}^{-\int_s^t b(u+r)\,\mathrm{d}u} \int_{s-r}^{s} |b(u+r)|\,\mathrm{d}u\,\mathrm{d}s \leqslant \alpha \tag{1.7}$$

for all $t \geqslant 0$ *and* $\int_0^\infty b(s)\,\mathrm{d}s = \infty$. *Then for every continuous initial function* $\psi : [-r, 0] \to R$, *the solution* $x(t) = x(t, 0, \psi)$ *of* (1.1) *is bounded and tends to zero as* $t \to \infty$.

Our aim here is to generalize Theorem D to (1.2) for unbounded $\tau_i(t)$'s and show that $\int_0^\infty b(s)\,\mathrm{d}s = \infty$ is a necessary condition for asymptotic stability.

2. Stability by contraction mapping

Let $R = (-\infty, \infty)$, $R^+ = [0, +\infty)$, and $R^- = (-\infty, 0]$, respectively. $C(S_1, S_2)$ denotes the set of all continuous functions $\phi : S_1 \to S_2$. For each t_0, define

$$m_j(t_0) = \inf\{s - \tau_j(s) : s \geqslant t_0\}, \quad m(t_0) = \min\{m_j(t_0) : 1 \leqslant j \leqslant N\},$$

and $C(t_0) = C([m(t_0), t_0], R)$ with the supremum norm $\|\cdot\|$. Define the inverse of $t - \tau_i(t)$ by $g_i(t)$ if it exists and set

$$Q(t) = \sum_{j=1}^{N} b_j(g_j(t)), \tag{2.1}$$

$$\theta(t) = \sum_{j=1}^{N} \int_0^t \mathrm{e}^{-\int_s^t Q(u)\,\mathrm{d}u} |b_j(s)||\tau_j'(s)|\,\mathrm{d}s \tag{2.2}$$

if $\tau_j(t)$ is differentiable. For each $(t_0, \phi) \in R^+ \times C(t_0)$, a solution of (1.2) through (t_0, ϕ) is a continuous function $x : [m(t_0), t_0 + \alpha) \to R^n$ for some positive constant $\alpha > 0$ such that $x(t)$ satisfies (1.2) on $[t_0, t_0 + \alpha)$ and $x(s) = \phi(s)$ for $s \in [m(t_0), t_0]$. We denote such a solution by $x(t) = x(t, t_0, \phi)$. For each $(t_0, \phi) \in R^+ \times C(t_0)$, there exists a unique solution $x(t) = x(t, t_0, \phi)$ of (1.2) defined on $[t_0, +\infty)$ (see [4]). For fixed t_0, we define

$$\|\phi\| = \max\{|\phi(s)| : m(t_0) \leqslant s \leqslant t_0\}.$$

Stability definitions may be found in [1,5], for example.

Theorem 2.1. *Suppose that τ_j is differentiable, the inverse function $g_j(t)$ of $t - \tau_j(t)$ exists, and there exists a constant $\alpha \in (0, 1)$ such that for $t \geqslant 0$*

(i)

$$\liminf_{t\to\infty} \int_0^t Q(s)\,\mathrm{d}s > -\infty,$$

(ii)

$$\sum_{j=1}^{N} \left[\int_{t-\tau_j(t)}^{t} |b_j(g_j(s))|\,\mathrm{d}s + \int_0^t \mathrm{e}^{-\int_s^t Q(u)\,\mathrm{d}u} |Q(s)| \times \int_{s-\tau_j(s)}^{s} |b_j(g_j(v))|\,\mathrm{d}v\,\mathrm{d}s \right] + \theta(t) \leqslant \alpha.$$

Then the zero solution of (1.2) *is asymptotically stable if and only if*

(iii)

$$\int_0^t Q(s)\,\mathrm{d}s \to \infty \quad \text{as } t \to \infty.$$

Proof. First, suppose that (iii) holds. For each $t_0 \geqslant 0$, we set

$$K = \sup_{t\geqslant 0} \{\mathrm{e}^{-\int_0^t Q(s)\,\mathrm{d}s}\}. \tag{2.3}$$

Let $\phi \in C(t_0)$ be fixed and define

$$S = \{x \in C([m(t_0), \infty), R) : x(t) \to 0 \text{ as } t \to \infty,$$
$$x(s) = \phi(s) \quad \text{for } s \in [m(t_0), t_0]\}. \tag{2.4}$$

Then S is a complete metric space with metric $\rho(x, y) = \sup_{t\geqslant t_0}\{|x(t) - y(t)|\}$.

Define $P: S \to S$ by $(Px)(t) = \phi(t)$ for $t \in [m(t_0), t_0]$ and

$$\begin{aligned}(Px)(t) = {} & \left(\phi(t_0) - \sum_{j=1}^{N} \int_{t_0-\tau_j(t_0)}^{t_0} b_j(g_j(s))\phi(s)\,\mathrm{d}s\right) \mathrm{e}^{-\int_{t_0}^{t} Q(u)\,\mathrm{d}u} \\ & + \sum_{j=1}^{N} \int_{t-\tau_j(t)}^{t} b_j(g_j(s))x(s)\,\mathrm{d}s \\ & - \int_{t_0}^{t} \mathrm{e}^{-\int_s^t Q(u)\,\mathrm{d}u} \sum_{j=1}^{N} b_j(s)\tau_j'(s)x(s-\tau_j(s))\,\mathrm{d}s \\ & - \int_{t_0}^{t} \mathrm{e}^{-\int_s^t Q(u)\,\mathrm{d}u} Q(s) \left(\sum_{j=1}^{N} \int_{s-\tau_j(s)}^{s} b_j(g_j(v))x(v)\,\mathrm{d}v\right) \mathrm{d}s \end{aligned} \tag{2.5}$$

for $t \geqslant t_0$. It is clear that $(Px) \in C([m(t_0), \infty), R)$. We now show that $(Px)(t) \to 0$ as $t \to \infty$. Since $x(t) \to 0$ and $t - \tau_j(t) \to \infty$ as $t \to \infty$, for each $\varepsilon > 0$, there exists a $T_1 > t_0$ such that $s \geqslant T_1$ implies $|x(s - \tau_j(s)| < \varepsilon$ for $j = 1, 2, \ldots, N$. Thus, for $t \geqslant T_1$, the last term I_4 in (2.5) satisfies

$$\begin{aligned} |I_4| = {} & \left| \int_{t_0}^{t} \mathrm{e}^{-\int_s^t Q(u)\,\mathrm{d}u} Q(s) \left(\sum_{j=1}^{N} \int_{s-\tau_j(s)}^{s} b_j(g_j(v))x(v)\,\mathrm{d}v\right) \mathrm{d}s \right| \\ \leqslant {} & \int_{t_0}^{T_1} \mathrm{e}^{-\int_s^t Q(u)\,\mathrm{d}u} |Q(s)| \left(\sum_{j=1}^{N} \int_{s-\tau_j(s)}^{s} |b_j(g_j(v))||x(v)|\,\mathrm{d}v\right) \mathrm{d}s \\ & + \int_{T_1}^{t} \mathrm{e}^{-\int_s^t Q(u)\,\mathrm{d}u} |Q(s)| \left(\sum_{j=1}^{N} \int_{s-\tau_j(s)}^{s} |b_j(g_j(v))||x(v)|\,\mathrm{d}v\right) \mathrm{d}s \\ \leqslant {} & \sup_{\sigma \geqslant m(t_0)} |x(\sigma)| \int_{t_0}^{T_1} \mathrm{e}^{-\int_s^t Q(u)\,\mathrm{d}u} |Q(s)| \left(\sum_{j=1}^{N} \int_{s-\tau_j(s)}^{s} |b_j(g_j(v))|\,\mathrm{d}v\right) \mathrm{d}s \\ & + \varepsilon \int_{T_1}^{t} \mathrm{e}^{-\int_s^t Q(u)\,\mathrm{d}u} |Q(s)| \left(\sum_{j=1}^{N} \int_{s-\tau_j(s)}^{s} |b_j(g_j(v))|\,\mathrm{d}v\right) \mathrm{d}s. \end{aligned}$$

By (iii), there exists $T_2 > T_1$ such that $t \geqslant T_2$ implies

$$\sup_{\sigma \geqslant m(t_0)} |x(\sigma)| \int_{t_0}^{T_1} \mathrm{e}^{-\int_s^t Q(u)\,\mathrm{d}u} |Q(s)| \left(\sum_{j=1}^{N} \int_{s-\tau_j(s)}^{s} |b_j(g_j(v))|\,\mathrm{d}v\right) \mathrm{d}s < \varepsilon.$$

Apply (ii) to obtain $|I_4| \leqslant \varepsilon + \varepsilon\alpha < 2\varepsilon$. Thus, $I_4 \to 0$ as $t \to \infty$. Similarly, we can show that the rest of the terms in (2.5) approach zero as $t \to \infty$. This yields $(Px)(t) \to 0$ as $t \to \infty$, and hence $Px \in S$. Also, by (ii), P is a contraction mapping with contraction constant α. By the Contraction Mapping Principle (Smart [8, p. 2]), P has a unique fixed point x in S

which is a solution of (1.2) with $x(s)=\phi(s)$ on $[m(t_0),t_0]$ and $x(t)=x(t,t_0,\phi)\to 0$ as $t\to\infty$.

To obtain the asymptotic stability, we need to show that the zero solution of (1.2) is stable. Let $\varepsilon>0$ be given and choose $\delta>0$ $(\delta<\varepsilon)$ satisfying $2\delta K \mathrm{e}^{\int_0^{t_0} Q(u)\,\mathrm{d}u}+\alpha\varepsilon<\varepsilon$. If $x(t)=x(t,t_0,\phi)$ is a solution of (1.2) with $\|\phi\|<\delta$, then $x(t)=(Px)(t)$ defined in (2.5). We claim that $|x(t)|<\varepsilon$ for all $t\geqslant t_0$. Notice that $|x(s)|<\varepsilon$ on $[m(t_0),t_0]$. If there exists $t^*>t_0$ such that $|x(t^*)|=\varepsilon$ and $|x(s)|<\varepsilon$ for $m(t_0)\leqslant s<t^*$, then it follows from (2.5) that

$$\begin{aligned}|x(t^*)|\leqslant{}&\|\phi\|\left(1+\sum_{j=1}^{N}\int_{t_0-\tau_j(t_0)}^{t_0}|b_j(g_j(s)|\,\mathrm{d}s\right)\mathrm{e}^{-\int_{t_0}^{t^*}Q(u)\,\mathrm{d}u}\\&+\varepsilon\sum_{j=1}^{N}\int_{t^*-\tau_j(t^*)}^{t^*}|b_j(g_j(s))|\,\mathrm{d}s\\&+\varepsilon\int_{t_0}^{t^*}\mathrm{e}^{-\int_s^{t^*}Q(u)\,\mathrm{d}u}\sum_{j=1}^{N}|b_j(s)||\tau_j'(s)|\,\mathrm{d}s\\&+\varepsilon\int_{t_0}^{t^*}\mathrm{e}^{-\int_s^{t^*}Q(u)\,\mathrm{d}u}|Q(s)|\left(\sum_{j=1}^{N}\int_{s-\tau_j(s)}^{s}|b_j(g_j(v))|\,\mathrm{d}v\right)\mathrm{d}s\\\leqslant{}&2\delta K\mathrm{e}^{\int_0^{t_0}Q(u)\,\mathrm{d}u}+\alpha\varepsilon<\varepsilon\end{aligned}\tag{2.6}$$

which contradicts the definition of t^*. Thus, $|x(t)|<\varepsilon$ for all $t\geqslant t_0$, and the zero solution of (1.2) is stable. This shows that the zero solution of (1.2) is asymptotically stable if (iii) holds.

Conversely, suppose (iii) fails. Then by (i) there exists a sequence $\{t_n\}$, $t_n\to\infty$ as $n\to\infty$ such that $\lim_{n\to\infty}\int_0^{t_n}Q(u)\,\mathrm{d}u=\ell$ for some $\ell\in R$. We may also choose a positive constant J satisfying

$$-J\leqslant\int_0^{t_n}Q(s)\,\mathrm{d}s\leqslant J$$

for all $n\geqslant 1$. To simplify expressions, we define

$$\omega(s)=\sum_{j=1}^{N}\left[|b_j(s)||\tau_j'(s)|+|Q(s)|\int_{s-\tau_j(s)}^{s}|b_j(g_j(v))|\,\mathrm{d}v\right]$$

for all $s\geqslant 0$. By (ii), we have

$$\int_0^{t_n}\mathrm{e}^{-\int_s^{t_n}Q(u)\,\mathrm{d}u}\omega(s)\,\mathrm{d}s\leqslant\alpha.$$

This yields

$$\int_0^{t_n}\mathrm{e}^{\int_0^{s}Q(u)\,\mathrm{d}u}\omega(s)\,\mathrm{d}s\leqslant\alpha\mathrm{e}^{\int_0^{t_n}Q(u)\,\mathrm{d}u}\leqslant\mathrm{e}^{J}.$$

The sequence $\{\int_0^{t_n} \mathrm{e}^{\int_0^s Q(u)\,\mathrm{d}u}\omega(s)\,\mathrm{d}s\}$ is bounded, so there exists a convergent subsequence. For brevity in notation, we may assume

$$\lim_{n\to\infty} \int_0^{t_n} \mathrm{e}^{\int_0^s Q(u)\,\mathrm{d}u}\omega(s)\,\mathrm{d}s = \gamma$$

for some $\gamma \in R^+$ and choose a positive integer $\bar{k}$ so large that

$$\int_{t_{\bar{k}}}^{t_n} \mathrm{e}^{\int_0^s Q(u)\,\mathrm{d}u}\omega(s)\,\mathrm{d}s < \delta_0/4K$$

for all $n \geqslant \bar{k}$, where $\delta_0 > 0$ satisfies $4\delta_0 K \mathrm{e}^J + \alpha < 1$.

By (i), K in (2.3) is well defined. We now consider the solution $x(t) = x(t, t_{\bar{k}}, \phi)$ of (1.2) with $\phi(t_{\bar{k}}) = \delta_0$ and $|\phi(s)| \leqslant \delta_0$ for $s \leqslant t_{\bar{k}}$. An argument similar to that in (2.6) shows $|x(t)| \leqslant 1$ for $t \geqslant t_{\bar{k}}$. We may choose ϕ so that

$$\phi(t_{\bar{k}}) - \sum_{j=1}^{N} \int_{t_{\bar{k}}-\tau_j(t_{\bar{k}})}^{t_{\bar{k}}} b_j(g_j(s))\phi(s)\,\mathrm{d}s \geqslant \frac{1}{2}\delta_0. \tag{2.7}$$

It follows from (2.5) with $x(t) = (Px)(t)$ that for $n \geqslant t_{\bar{k}}$,

$$\begin{aligned}
&\left| x(t_n) - \sum_{j=1}^{N} \int_{t_n-\tau_j(t_n)}^{t_n} b_j(g_j(s))x(s)\,\mathrm{d}s \right| \\
&\quad\geqslant \frac{1}{2}\delta_0 \mathrm{e}^{-\int_{t_{\bar{k}}}^{t_n} Q(u)\,\mathrm{d}u} - \int_{t_{\bar{k}}}^{t_n} \mathrm{e}^{-\int_s^{t_n} Q(u)\,\mathrm{d}u}\omega(s)\,\mathrm{d}s \\
&\quad= \frac{1}{2}\delta_0 \mathrm{e}^{-\int_{t_{\bar{k}}}^{t_n} Q(u)\,\mathrm{d}u} - \mathrm{e}^{-\int_0^{t_n} Q(u)\,\mathrm{d}u} \int_{t_{\bar{k}}}^{t_n} \mathrm{e}^{\int_0^s Q(u)\,\mathrm{d}u}\omega(s)\,\mathrm{d}s \\
&\quad= \mathrm{e}^{-\int_{t_{\bar{k}}}^{t_n} Q(u)\,\mathrm{d}u} \left[\frac{1}{2}\delta_0 - \mathrm{e}^{-\int_0^{t_{\bar{k}}} Q(u)\,\mathrm{d}u} \int_{t_{\bar{k}}}^{t_n} \mathrm{e}^{\int_0^s Q(u)\,\mathrm{d}u}\omega(s)\,\mathrm{d}s \right] \\
&\quad\geqslant \mathrm{e}^{-\int_{t_{\bar{k}}}^{t_n} Q(u)\,\mathrm{d}u} \left[\frac{1}{2}\delta_0 - K \int_{t_{\bar{k}}}^{t_n} \mathrm{e}^{\int_0^s Q(u)\,\mathrm{d}u}\omega(s)\,\mathrm{d}s \right] \\
&\quad\geqslant \frac{1}{4}\delta_0 \mathrm{e}^{-\int_{t_{\bar{k}}}^{t_n} Q(u)\,\mathrm{d}u} \geqslant \frac{1}{4}\delta_0 \mathrm{e}^{-2J} > 0.
\end{aligned} \tag{2.8}$$

On the other hand, if the zero solution of (1.2) is asymptotically stable, then $x(t) = x(t, t_{\bar{k}}, \phi) \to 0$ as $t \to \infty$. Since $t_n - \tau_j(t_n) \to \infty$ as $n \to \infty$ and (ii) holds, we have

$$x(t_n) - \sum_{j=1}^{N} \int_{t_n-\tau_j(t_n)}^{t_n} b_j(g_j(s))x(s)\,\mathrm{d}s \to 0 \quad \text{as } t \to \infty$$

which contradicts (2.8). Hence, condition (iii) is necessary for the asymptotically stability of the zero solution of (1.2). The proof is complete. □

Remark 2.1. It follows from the first part of the proof of Theorem 2.1 that the zero solution of (1.2) is stable under (i) and (ii). Moreover, Theorem 2.1 still holds if (ii) is satisfied for $t \geqslant t_\sigma$ for some $t_\sigma \in R^+$.

Remark 2.2. If $Q(t) \geqslant 0$ and $\theta(t) \leqslant \ell$ for $\ell \in [0, 1)$, then (i) is satisfied and (ii) can be replaced by

$$\sup_{t \geqslant 0} \sum_{j=1}^{N} \int_{t-\tau_j(t)}^{t} |b_j(g_j(s))| \,\mathrm{d}s < (1-\ell)/2.$$

Corollary 2.1. *Let* $N = 1$ *and suppose that* τ *is differentiable, the inverse function* $g(t)$ *of* $t - \tau(t)$ *exists, and there exists a constant* $\alpha \in (0, 1)$ *such that for* $t \geqslant 0$

(i*)

$$\liminf_{t \to \infty} \int_0^t b(g(s)) \,\mathrm{d}s > -\infty,$$

(ii*)

$$\int_{t-\tau(t)}^{t} |b(g(s))| \,\mathrm{d}s + \int_0^t \mathrm{e}^{-\int_s^t b(g(u))\,\mathrm{d}u} |b(g(s))| \int_{s-\tau(s)}^{s} |b(g(v))| \,\mathrm{d}v \,\mathrm{d}s + \theta(t) \leqslant \alpha,$$

where $\theta(t) = \int_0^t \mathrm{e}^{-\int_s^t b(g(u))\,\mathrm{d}u} |b(s)||\tau'(s)| \,\mathrm{d}s$. *Then the zero solution of* (1.1) *is asymptotically stable if and only if*

(iii*)

$$\int_0^t b(g(s)) \,\mathrm{d}s \to \infty, \quad \text{as } t \to \infty.$$

3. An example

In this section, we give an example to illustrate how to apply Theorem 2.1 to a specific equation. The example is in a simple form for illustrative purpose, and it can easily be generalized. We are not seeking the best estimate for the constants involved here.

Example 3.1. Consider the scalar equation

$$x'(t) = -b_1(t)x(t - \tau_1(t)) - b_2(t)x(t - \tau_2(t)), \tag{3.1}$$

where $\tau_1(t) = \mu(2t - \sin t)$, $\tau_2(t) = 2\mu t$, and

$$b_1(t) = \frac{\gamma}{1+t}, \quad b_2(t) = \frac{2\gamma \sin t}{1+t}.$$

To be specific, let $\mu = 0.0025$ and $\gamma = 0.4$. Then the zero solution of (3.1) is asymptotically stable.

Proof. The functions $t-\tau_j(t)$, $j=1,2$, are increasing with $t-\tau_j(t)\to\infty$ as $t\to\infty$. We denote their inverses by $g_j(t)$, respectively. It is clear that $g_2(u)=u/(1-2\mu)$. Since

$$f_1(t)=:(1-2\mu)t-1\leqslant t-\tau_1(t)\leqslant(1-\mu)t=:f_2(t),$$

we have $f_2^{-1}(u)\leqslant g_1(u)\leqslant f_1^{-1}(u)$, that is, $u/(1-\mu)\leqslant g_1(u)\leqslant(u+1)/(1-2\mu)$ for all $u\geqslant 0$. Observe that

$$\int_{t-\tau_1(t)}^{t}|b_1(g_1(s))|\,\mathrm{d}s\leqslant\int_{(1-2\mu)t-\mu}^{t}\frac{\gamma}{1+s/(1-\mu)}\,\mathrm{d}s\leqslant\gamma(1-\mu)|\ln(1-2\mu)|$$

and

$$\int_{t-\tau_2(t)}^{t}|b_2(g_2(s))|\,\mathrm{d}s\leqslant\int_{(1-2\mu)t}^{t}\frac{2\gamma}{1+s/(1-2\mu)}\,\mathrm{d}s\leqslant 2\gamma(1-2\mu)|\ln(1-2\mu)|.$$

Thus,

$$\sum_{j=1}^{2}\int_{t-\tau_j(t)}^{t}|b_j(g_j(s))|\,\mathrm{d}s\leqslant\gamma(3-5\mu)|\ln(1-2\mu)|\leqslant 0.006. \tag{3.2}$$

Use the fact that $|\int_a^b \sin v/(1+v)\,\mathrm{d}v|\leqslant 3$ for $b\geqslant a\geqslant 0$ to obtain

$$\begin{aligned}-\int_s^t Q(u)\,\mathrm{d}u&=-\int_s^t\frac{\gamma}{1+g_1(u)}\,\mathrm{d}u-\int_s^t\frac{2\gamma\sin g_2(u)}{1+g_2(u)}\,\mathrm{d}u\\&\leqslant-\int_s^t\frac{\gamma}{1+g_1(u)}\,\mathrm{d}u+6\gamma(1-2\mu).\end{aligned}\tag{3.3}$$

Notice that

$$|Q(t)|\leqslant\frac{\gamma}{1+g_1(t)}+\frac{2\gamma|\sin(g_2(t))|}{1+g_2(t)}\leqslant\frac{\gamma(3-5\mu)}{(1-2\mu)+t}.$$

Since $(a+1+t)/(a+t)$ is decreasing in $t\in R^+$ for $a\geqslant 0$, we have

$$\begin{aligned}|Q(t)|&\leqslant\frac{(1-2\mu)+1+t}{(1-2\mu)+t}\,\frac{\gamma(3-5\mu)}{(1-2\mu)+1+t}\\&\leqslant\frac{(1-2\mu)+1}{(1-2\mu)}\,\frac{\gamma(3-5\mu)}{(1-2\mu)+1+t}\leqslant\frac{2(1-\mu)}{(1-2\mu)^2}\,\frac{\gamma(3-5\mu)}{1+g_1(t)}.\end{aligned}\tag{3.4}$$

Using (3.2)–(3.4), we get

$$\begin{aligned}
&\int_0^t \mathrm{e}^{-\int_s^t Q(u)\,\mathrm{d}u}|Q(s)|\left(\sum_{j=1}^{2}\int_{s-\tau_j(s)}^{s}|b_j(g_j(v))|\,\mathrm{d}v\right)\mathrm{d}s\\
&\leqslant\gamma(3-5\mu)|\ln(1-2\mu)|\int_0^t \mathrm{e}^{-\int_s^t Q(u)\,\mathrm{d}u}|Q(s)|\,\mathrm{d}s\\
&\leqslant\gamma(3-5\mu)|\ln(1-2\mu)|e^{6\gamma}\frac{2(1-\mu)(3-5\mu)}{(1-2\mu)^2}\int_0^t \mathrm{e}^{-\int_s^t \frac{\gamma}{1+g_1(u)}\,\mathrm{d}u}\frac{\gamma}{1+g_1(s)}\,\mathrm{d}s\\
&\leqslant\gamma(3-5\mu)|\ln(1-2\mu)|e^{6\gamma}\frac{2(1-\mu)(3-5\mu)}{(1-2\mu)^2}<0.398. \qquad (3.5)
\end{aligned}$$

Taking into account the fact that $|\tau_j'(t)|\leqslant 3\mu$, we arrive at

$$\int_0^t \mathrm{e}^{-\int_s^t Q(u)\,\mathrm{d}u}\sum_{j=1}^{2}|b_j(s)||\tau_j'(s)|\,\mathrm{d}s\leqslant 3\mu\mathrm{e}^{6\gamma}\frac{6(1-\mu)}{1-2\mu}<0.498. \qquad (3.6)$$

Combining (3.2), (3.5), and (3.6), we see that condition (ii) of Theorem 2.1 holds with $\alpha=0.902$. One can show that $\int_0^\infty Q(s)\,\mathrm{d}s=\infty$. Thus, the asymptotic stability of the zero solution of (3.1) follows from Theorem 2.1. □

Remark 3.1. More examples of stability on (1.1) by fixed point theory can be found in Burton [2,3] and references therein.

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